help explain why the product of two negative real numbers must be positive (recall that multiplication of complex numbers involves a rotation). Indeed, the shortest path between two truths in the real domain does pass through the complex domain.

References

1. William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations*, 6th ed., Wiley, New York, 1997.
2. Ruel V. Churchill and J. W. Brown, *Complex Variables and Applications*, 5th ed., McGraw-Hill, New York, 1990.
3. Victor J. Katz, *A History of Mathematics*, Harper Collins, New York, 1993.

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Polishing the Star

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Recently, Hoehn proved the following interesting theorem about a pentagram [A Menelaus-type theorem for the pentagram, *Mathematics Magazine* 66:2 121–123] Hoehn used Menelaus' theorem 20 times, but it is possible to give a much simpler proof. Geometry students may enjoy seeing results concerning the pentagram as an application of the Law of Sines.

Theorem. *If* $A_1B_1A_2B_2A_3B_3A_4B_4A_5B_5$ *is a pentagram* (see Figure 1), *then*

$$\frac{A_1B_1}{B_1A_2}\cdot\frac{A_2B_2}{B_2A_3}\cdot\frac{A_3B_3}{B_3A_4}\cdot\frac{A_4B_4}{B_4A_5}\cdot\frac{A_5B_5}{B_5A_1}=1 \tag{1}$$

and

$$\frac{B_1A_3}{A_3B_4}\cdot\frac{B_4A_1}{A_1B_2}\cdot\frac{B_2A_4}{A_4B_5}\cdot\frac{B_5A_2}{A_2B_3}\cdot\frac{B_3A_5}{A_5B_1}=1. \tag{2}$$

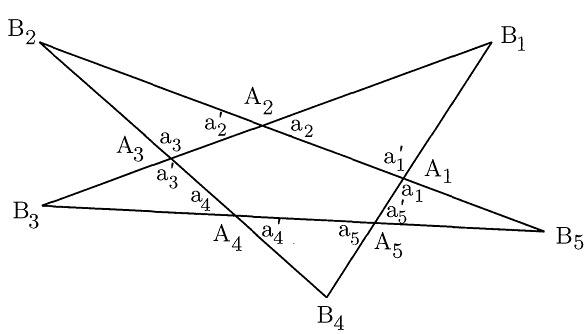

Figure 1

Proof. To obtain (1) we use the Law of Sines in the five triangles $A_1B_1A_2$, $A_1B_2A_3$, $A_3B_3A_4$, $A_4B_4A_5$, and $A_5B_5A_1$, finding

$$\frac{A_1B_1}{B_1A_2} = \frac{\sin a_2}{\sin a_1'}, \quad \frac{A_2B_2}{B_2A_3} = \frac{\sin a_3}{\sin a_2'}, \quad \frac{A_3B_3}{B_3A_4} = \frac{\sin a_4}{\sin a_3'},$$
$$\frac{A_4B_4}{B_4A_5} = \frac{\sin a_5}{\sin a_4'}, \quad \text{and} \quad \frac{A_5B_5}{B_5A_1} = \frac{\sin a_1}{\sin a_5'},$$

respectively. Since $a_k = a_k'$ for each $k = 1, \ldots, 5$, we can multiply these five equations; (1) is the result. Equation (2) is obtained in a similar way by using the Law of Sines in the five triangles $B_1A_3B_4$, $B_4A_1B_2$, $B_2A_4B_5$, $B_5A_2B_3$, and $B_3A_5B_1$. □

Here is another simple fact about the pentagram that may be surprising and appealing to the beginning student: *The sum of the angles at the points of the star is* 180°. One way to see this is to observe that $a_1' = \angle B_2 + \angle B_4$, $a_2 = \angle B_3 + \angle B_5$, and $\angle B_1 + a_1' + a_2 = 180°$ and then compute $\angle B_1 + \angle B_2 + \angle B_3 + \angle B_4 + \angle B_5$. This same result can be derived from the fact that the sum of the exterior angles of any convex n-gon is 360°. Thus, the five triangles containing the points of the star have $a_1 + a_2 + a_3 + a_4 + a_5 = 360°$ and $a_1' + a_2' + a_3' + a_4' + a_5' = 360°$. This leaves $5(180°) - 360° - 360° = 180°$ for $\angle B_1 + \angle B_2 + \angle B_3 + \angle B_4 + \angle B_5$.

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When Is "Rank" Additive?

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Most matrix theory books mention that rank is subadditive—that is, rank $(A + B) \leq$ rank $(A)+$ rank (B)—but they rarely address the question of equality. Recall that the rank of a matrix A is defined as the *dimension of its column space* $C(A)$. Also, the rank is invariant under transpose: rank (A) = rank (A^T); or, what is the same, the rank of A is the dimension of the row space $R(A)$. (See [**2**] and [**3**] for one-paragraph proofs of this fundamental fact.) This leads to a useful alternative description of the rank: *Rank* (A) *is the size of the largest invertible submatrix of* A.

The subadditivity of rank is easily established: $C(A + B) \subseteq C(A) + C(B)$, hence rank $(A+B) = \dim C(A+B) \leq \dim[C(A)+C(B)] \leq \dim C(A) + \dim C(B) =$ rank $(A)+$ rank (B). Since $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$ for any two subspaces U and V, equality in the second inequality above implies $C(A) \cap C(B) = \{0\}$. Thus disjointness of the column spaces of A and B is a necessary condition for additivity of rank. Curiously, a recent monograph [**4**] asserts incorrectly that this condition is sufficient.

Counterexample. $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $C(A) \cap C(B) = \{0\}$, but rank (A) = rank (B) = rank $(A + B) = 1$).

However another necessary condition is disjointness of the row spaces (since rank is invariant under transpose). It turns out that these two conditions together are sufficient.