The Factorial Triangle and Polynomial Sequences

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In the March 1980 Classroom Capsules Column, Kenneth Kundert discusses synthetic multiplication for

$$(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)(x + k).$$

Specifically:

$$\begin{array}{c|ccccccccccc}
k & a_n & & a_{n-1} & & a_{n-2} & & \cdots & & a_1 & & a_0 \\
 & & & ka_n & & ka_{n-1} & & ka_{n-2} & & ka_2 & & ka_1 & & ka_0 \\
\hline
 & a_n & & a_{n-1} + ka_n & & a_{n-2} + ka_{n-1} & \cdots & & & a_1 + ka_2 & & a_0 + ka_1 & & ka_0
\end{array},$$

where the entries in the third row are the coefficients of the resulting $(n + 1)$st degree product polynomial. As an application, Kundert illustrates how repeated synthetic multiplication of $(x + 1)$ by $(x + 1)$ generates the binomial coefficients. Thus, if the entries in the kth row of Pascal's Triangle are recognized as the coefficients of the polynomial $(x + 1)^k$, the first row being the Oth, then the entries in the $(k + 1)$st row are the coefficients of the product $(x + 1)^k \cdot (x + 1)$.

We can extend this idea by introducing the following triangular array, calling it the *Factorial Triangle*.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 2 & & & \\
 & & 1 & & 6 & & 11 & & 6 & & \\
 & 1 & & 10 & & 35 & & 50 & & 24 & \\
1 & & 15 & & 85 & & 225 & & 274 & & 120 \\
 & & & & & \text{etc.} & & & & &
\end{array}$$

Students already familiar with Pascal's Triangle usually enjoy the challenge of trying to figure out this more complicated pattern and thereby generate some more rows. Although some students have noticed that the numbers down the right side of the triangle are the factorials of the row numbers (assuming again that the first row is row 0), and others have noticed that $(k + 1)!$ is the sum of the entries in row k, the secret of the Factorial Triangle has eluded almost everyone: if row k represents the coefficients of a polynomial, then row $(k + 1)$ represents the coefficients of the product when that polynomial is multiplied by $[x + (k + 1)]$. Thus, for example, the entry 35 in row 4 results from multiplying the 6 in row 3 by 4 and adding the 11 next to it. This process of multiplying and adding the next number is perfectly general, but the multiplier increases by 1 from each row to the next. If row 0 represents the coefficient of x, then row 1 represents the coefficients of $x(x + 1)$; row 2 represents the coefficients of $x(x + 1)(x + 2)$; and so on.

This pattern is more than just a curiosity. It is related to finding the formula for a generating polynomial when enough of its consecutive numerical values are given. The method involves consecutive differences as illustrated:

Beginning with the sequence	a_1		a_2		a_3		$a_4 \cdots,$
the first differences are		b_1		b_2		b_3	$\cdots,$
the second differences are			c_1		c_2		$\cdots,$
and so on.							

Upon closer examination of the a_i, we see that

$$a_1 = a_1$$

$$a_2 = a_1 + b_1$$

$$a_3 = a_2 + b_2 = (a_1 + b_1) + (b_1 + c_1) = a_1 + 2b_1 + c_1$$

$$a_4 = a_3 + b_3 = (a_1 + 2b_1 + c_1) + (b_2 + c_2) = a_1 + 3b_1 + 3c_1 + d_1 .$$

The pattern is now clear: a_n is a sum of terms whose coefficients are the entries in row $n - 1$ of Pascal's Triangle. But it is well known that the ith entry in row k of Pascal's Triangle is given by

$$\binom{k}{i} = \frac{k(k-1)(k-2)\cdots(k-i+1)}{i!} . \tag{1}$$

As a result, the formula for the general term of any sequence generated by a polynomial is

$$a_n = \binom{n-1}{0}a_1 + \binom{n-1}{1}b_1 + \binom{n-1}{2}c_1 + \binom{n-1}{3}d_1 + \cdots$$

$$= a_1(1) + \frac{b_1}{1!}(n-1) + \frac{c_1}{2!}(n^2 - 3n + 2) + \frac{d_1}{3!}(n^3 - 6n^2 + 11n - 6) + \cdots .$$

At this point the role of the entries in the Factorial Triangle becomes apparent: the coefficients of each polynomial factor are precisely the rows of the Factorial Triangle, except that the plus and minus signs alternate.

As an example, consider the following sequence and its differences:

a_1	=	10		11		0		-29		-82	$\cdots$	
	b_1	=	1		-11		-29		-53	$\cdots$		
		c_1	=	-12		-18		-24	$\cdots$			
			d_1	=	-6		-6	$\cdots$				
						0						

Here $a_n = 10(1) + 1(n-1) - \frac{12}{2}(n^2 - 3n + 2) - \frac{6}{6}(n^3 - 6n^2 + 11n - 6)$, and, upon simplification, we see that $a_n = -n^3 + 8n + 3$ does indeed generate the initial sequence for $n = 1, 2, 3, 4, 5$.

The Factorial Triangle is curiously intertwined with Pascal's Triangle. When trying to find the polynomial formulas that generate each diagonal of Pascal's Triangle, one can apply the method of differences. It is surprising to see that a_1, b_1, c_1, etc. (the first entries in the rows of differences) are nothing more than rows of Pascal's Triangle, though in retrospect it's not hard to see why that must be so: it's

because each entry in Pascal's Triangle is the sum of the two entries that "straddle" it in the previous row, and so the process of taking successive differences starting on a *diagonal* leads back to the appropriate *row*. The entries in the ith diagonal of Pascal's Triangle, then, are given by

$$d_i(k) = \frac{k(k+1)(k+2)(k+3)\cdots(k+i-1)}{i!}. \tag{2}$$

Notice the striking similarity of equations (1) and (2): the only difference is that the signs within each factor are reversed. If those factors are multiplied out for each value of i, the coefficients of the resulting polynomials are the entries in the rows of the Factorial Triangle.

Editor's Note: Readers interested in a fuller exposition on sequences generated by polynomials may enjoy Calvin Long's article "Pascal's Triangle, Difference Tables, and Arithmetic Sequences of Order n," CMJ 15 (September 1984) 290–298.

Finding Bounds for Definite Integrals

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Students in elementary calculus are often dismayed to learn that not every function has an antiderivative, and consequently not every definite integral can be evaluated by the Fundamental Theorem. Although most textbooks discuss such things as Simpson's Rule and the Trapezoid Rule, these methods are usually long and tedious to apply. In many cases, reasonably good bounds for definite integrals can be obtained with little effort by the use of well-known theorems. The fact that techniques for doing this have never been discussed in one place is the motivation for this note.

Except for very specialized and esoteric results, the following three theorems provide methods for obtaining such bounds.

Theorem A. *If f, g, and h are integrable and satisfy $g(x) \leqslant f(x) \leqslant h(x)$ on the interval $[a,b]$, then*

$$\int_a^b g(x)\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b h(x)\,dx.$$

Theorem B. *On the interval $[a,b]$, suppose that f and g are integrable, g never changes sign, and $m \leqslant f(x) \leqslant M$. Then*

$$m\int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant M\int_a^b g(x)\,dx.$$

Theorem C. *If f and g are integrable on $[a,b]$, then*

$$\int_a^b f(x)g(x)\,dx \leqslant \sqrt{\int_a^b f^2(x)\,dx}\,\sqrt{\int_a^b g^2(x)\,dx}\,.$$