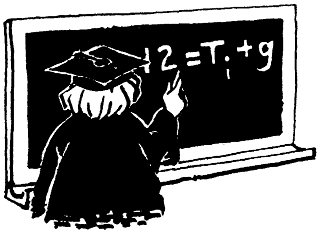

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

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Approximate Angle Trisection

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In the March 1983 Classroom Capsules Column, G. Peterson discusses H. Steinhaus' compass and straightedge approximation for trisecting any given angle A of measure α (Figure 1): *First bisect A and then trisect chord BE. The desired approximation was angle DAB having measure t.* In this note, we present a simpler construction which gives a more accurate approximation. Just as Peterson's analysis provided students with exercises in trigonometry and the use of a calculator, so does the present discussion; in fact, we go even further and illustrate how the Taylor series can be used to estimate, and therefore compare, the error in each of these approximations.

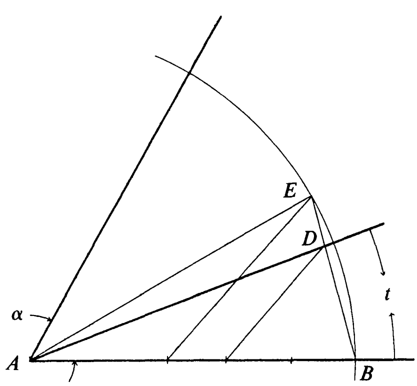

Figure 1.

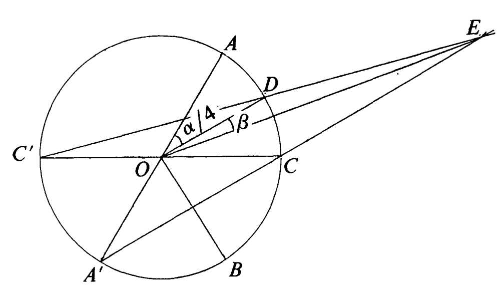

Figure 2.

H. W. Segar's "Trisection of an Angle" [Trans. N. Z. Institute, 41(1908) 218–221] is the earliest reference I've been able to find for the present construction (Figure 2): *Bisect angle AOB having measure α and then bisect angle AOC of measure $\alpha/2$. Let $A'C$ and $C'D$ meet at E. Then the desired approximation is angle AOE having measure θ.*

Clearly $\sphericalangle OA'E = \frac{1}{2}\sphericalangle AOC = \alpha/4$, and $\sphericalangle A'EO = \beta$ since OD is parallel to $A'E$ (note that $\sphericalangle DOC = \sphericalangle OCA'$). Thus, the law of sines applied to triangle $A'OE$ yields

$$\frac{\sin(\alpha/4)}{OE} = \frac{\sin\beta}{A'O}. \tag{1}$$

In triangle $C'OE$, we see that $\sphericalangle OC'E = \frac{1}{2}(\sphericalangle COD) = \alpha/8$ and, since an exterior angle of a triangle is the sum of the remote interior angles, we also have $\sphericalangle C'EO = \sphericalangle COE - \sphericalangle OC'E$. Hence $\sphericalangle C'EO = [(\alpha/4) - \beta] - (\alpha/8) = (\alpha/8) - \beta$. Therefore, the law of sines in triangle $C'OE$ yields

$$\frac{\sin(\alpha/8)}{OE} = \frac{\sin((\alpha/8) - \beta)}{C'O}. \tag{2}$$

Together, (1) and (2) give

$$\sin\frac{\alpha}{8}\sin\beta = \sin\frac{\alpha}{4}\sin\left(\frac{\alpha}{8} - \beta\right),$$

from which it readily follows that

$$\tan\beta = \frac{\sin\frac{\alpha}{4}}{2 + \cos\frac{\alpha}{4}}.$$

Since $\theta = \beta + \alpha/4$, we obtain

$$\tan\theta = \frac{\sin(\alpha/2) + 2\sin(\alpha/4)}{\cos(\alpha/2) + 2\cos(\alpha/4)}. \tag{3}$$

When α (measured in radians) is small we see, as in Peterson's reasoning, that $\theta \approx \alpha/3$. Now let us compare (3) with Peterson's result

$$\tan t = \frac{2\sin(\alpha/2)}{1 + 2\cos(\alpha/2)}.$$

Using Taylor series expansions for sin and cos, up to terms of degree 3, we obtain

$$\sin(\alpha/2) + 2\sin(\alpha/4) \approx \frac{\alpha}{2} - \frac{\alpha^3}{48} + 2\left(\frac{\alpha}{4} - \frac{\alpha^3}{384}\right) = \alpha - \frac{5\alpha^3}{192}$$

and

$$\cos(\alpha/2) + 2\cos(\alpha/4) \approx 1 - \frac{\alpha^2}{8} + 2\left(1 - \frac{\alpha^2}{32}\right) = 3\left(1 - \frac{\alpha^2}{16}\right).$$

Thus, up to terms of degree 3, (3) can be recast as

$$\tan\theta \approx \left(\alpha - \frac{5\alpha^3}{192}\right)\left[3\left(1 - \frac{\alpha^2}{16}\right)\right]^{-1}$$
$$\approx \frac{1}{3}\left(\alpha - \frac{5\alpha^3}{192}\right)\left(1 + \frac{\alpha^2}{16}\right)$$
$$\approx \frac{\alpha}{3} + \frac{7\alpha^3}{576}.$$

Finally, by applying Taylor series expansion for arctan we obtain

$$\theta \approx \arctan\left(\frac{\alpha}{3} + \frac{7\alpha^3}{576}\right)$$
$$\approx \frac{\alpha}{3} + \frac{7\alpha^3}{576} - \frac{\alpha^3}{81}$$
$$\approx \frac{\alpha}{3} - \frac{\alpha^3}{5184}.$$

Similar calculations using Peterson's result yields

$$t \approx \frac{\alpha}{3} + \frac{\alpha^3}{648}.$$

Thus, although both θ and t approximate $\alpha/3$, we see that θ gives the better approximation. This could be confirmed by use of a calculator.

Alternate Approaches to Two Familiar Results

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It is not particularly easy to prove that $\sqrt[n]{n!}$ becomes infinitely large as n increases or, more exactly,

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \tag{1}$$

The standard proof of (1) involves Stirling's formula (see, for example, A. Taylor, *Advanced Calculus*, Ginn & Co. (1955), p. 684), and so it is rarely included in a first course in calculus.

In this note, we offer a simple proof of (1) that is based on the familiar double inequality

$$\left(1 + \frac{1}{k}\right)^k < e < \left(1 + \frac{1}{k}\right)^{k+1} \tag{2}$$

for all positive integers k. We first give a proof of (2) that rests only on the theorem of the Mean for Derivatives. Thus, (1) and (2) can be presented early in a first course in calculus.