Here are some examples of the use of the definition. Suppose that for a current period of my life (from a_1 to a_2) I want to compute the age which is its perceptual midpoint. That is, I want to find a_m which satisfies $L(a_1, a_m) = L(a_m, a_2)$. We find that $a_m = \sqrt{a_1 a_2}$. If I use for a_1 the age of my earliest memory (for me about 5), then I can find the age at which my life from then to my present age, 31.33 (31 years and 121 days) was, relatively speaking, at its halfway point: $a_m = \sqrt{5 \cdot 31.33} = 12.52$.

Suppose that I want to know when my life will reach its perceptual halfway point. If I assume that I'll live to age 80, and again using age 5 as my starting age, the halfway point in my life is exactly age 20. So, according to the model, my life is already more than half over. In fact,

$$\frac{L(31.33, 80)}{L(5, 80)} = \frac{\ln(80/31.33)}{\ln(80/5)} = 0.34$$

so it might seem that I have only about a third of my life yet to live.

For a person of a certain age, the perceived length of the coming year relative to the perceived length of the previous year is

$$N(a) = \frac{L(a, a+1)}{L(a-1, a)} = \frac{\ln((a+1)/a)}{\ln(a/(a-1))}.$$

So, the day after Christmas, when my daughter poses the "How long until next Christmas?" question I can do more than tell her "One year." I can use our ages—she will be 5.21 and I will be 31.48—to find the answer: since $N(5.21) = 0.82$ and $N(31.48) = 0.97$, I could tell her, "For you it will seem to take about 82% as long until next Christmas as it took to get to this Christmas and for me it will seem to take about 97% as long." Then she'll ask, "Daddy, what does 'percent' mean?"

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Sum Rearrangements

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Although it is known that the terms of a conditionally convergent series can be rearranged to form a series that converges to any given real number (see Rudin [**3**, p. 76]), in general it is difficult to find the sum of a given rearrangement. We will show that, for an entire class of series and type of rearrangement, the sum of the rearrangement is easy to find.

Let a and b be real numbers such that both a and $a + b$ are positive, and consider the conditionally convergent series

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{ai + b}.$$

Let $\{s_n\}$ be its sequence of partial sums and let S be its sum. Let p and q be positive integers with $p > q$. Rearrange the series by taking the first p positive terms of the original series followed by the first q negative terms, then the next p positive terms, the next q negative terms, and so on. For example, with $a = 2$, $b = -1$, $p = 3$, and $q = 2$, the original series is

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\frac{1}{21}-\frac{1}{23}+\frac{1}{25}-\frac{1}{27}+\cdots$$

and the rearranged series is

$$1+\frac{1}{5}+\frac{1}{9}-\frac{1}{3}-\frac{1}{7}+\frac{1}{13}+\frac{1}{17}+\frac{1}{21}-\frac{1}{11}-\frac{1}{15}+\frac{1}{25}+\frac{1}{29}+\frac{1}{33}-\frac{1}{19}-\frac{1}{23}+\cdots.$$

We will prove that the sum of the rearranged series is $S+\ln(p/q)/(2a)$.

Lemma 1. *If $z_n=\sum_{i=qn+1}^{pn}(\frac{1}{2ai+b}-\frac{1}{2ai})$, then $\{z_n\}$ converges to* 0.

Proof. For each positive integer n,

$$|z_n|\le\sum_{i=qn+1}^{pn}\frac{|b|}{(2ai)(2ai+b)}\le\sum_{i=qn+1}^{pn}\frac{|b|}{(2ai)\big(2a(i-1)\big)}$$
$$\le\sum_{i=qn+1}^{pn}\frac{|b|}{(2aqn)^2}=\frac{|b|}{4a^2q^2n^2}(p-q)n=\frac{c}{n},$$

where c is a positive constant, implying the desired result.

Lemma 2. *If $x_n=\sum_{i=qn+1}^{pn}\frac{1}{i}$, then $\{x_n\}$ converges to* $\ln(p/q)$.

Proof. One way to prove this result is to use the fact that the sequence $\{y_n\}$, where

$$y_n=\sum_{i=1}^{n}\frac{1}{i}-\int_1^n\frac{dx}{x},$$

converges. (It is a decreasing sequence of positive numbers.) Since

$$x_n=\left(y_{pn}+\int_1^{pn}\frac{dx}{x}\right)-\left(y_{qn}+\int_1^{qn}\frac{dx}{x}\right)=y_{pn}-y_{qn}+\ln(p/q)$$

for each positive integer n, the sequence $\{x_n\}$ converges to $\ln(p/q)$.

Lemma 3. *Let $\sum_{i=1}^{\infty}d_i$ be a series for which $\lim_{i\to\infty}d_i=0$, let $\{t_n\}$ be its sequence of partial sums, and let j be a fixed positive integer. If the sequence $\{t_{jn}\}$ converges to t, then $\{t_n\}$ converges to t.*

Proof. Let $\epsilon>0$. Since $\lim_{i\to\infty}d_i=0$, there exists a positive integer N_1 such that $|d_i|<\epsilon/j$ for all $i\ge N_1$. Since $\{t_{jn}\}$ converges to t, there exists an integer $N>N_1$ such that $|t_{jn}-t|<\epsilon$ for all $n\ge N$. Suppose that $n\ge jN$. Then there exists an integer $m\ge N$ such that $mj\le n<(m+1)j$ and it follows that

$$|t_n-t|\le|t_n-t_{jm}|+|t_{jm}-t|\le\sum_{i=mj+1}^{(m+1)j}|d_i|+|t_{jm}-t|<j\cdot\frac{\epsilon}{j}+\epsilon=2\epsilon.$$

This shows that $|t_n-t|<2\epsilon$ for all $n\ge jN$. Therefore, the sequence $\{t_n\}$ converges to t.

We can now prove the main result. Let $\{r_n\}$ be the sequence of partial sums of the rearranged series. For each positive integer n,

$$\begin{aligned}
r_{(p+q)n} &= \sum_{i=1}^{pn} \frac{1}{a(2i-1)+b} - \sum_{i=1}^{qn} \frac{1}{a(2i)+b} \\
&= \sum_{i=1}^{pn} \frac{1}{a(2i-1)+b} - \sum_{i=1}^{pn} \frac{1}{a(2i)+b} + \sum_{i=qn+1}^{pn} \left(\frac{1}{2ai+b} - \frac{1}{2ai} \right) + \sum_{i=qn+1}^{pn} \frac{1}{2ai} \\
&= s_{2pn} + z_n + \frac{1}{2a} x_n,
\end{aligned}$$

where the terms s_n, z_n, and x_n have been defined previously. By Lemmas 1 and 2, the sequence $\{r_{(p+q)n}\}$ converges to $S + \ln(p/q)/(2a)$. The result then follows from Lemma 3.

We conclude this paper with several observations.

1. Many different rearrangements will give the same sum; the sum of the rearranged series has the same value as long as p/q is constant.
2. The reader can check that the same formula for the rearranged sum is valid for the case in which $q > p$. (Note that $\ln(p/q)$ is a negative number in this case.) The computations for $r_{(p+q)n}$ are slightly different, but no new ideas are involved.
3. For most of the series considered in this paper, it is not possible to find an exact value for S. However, using the geometric series, it can be shown that

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{ai + (1-a)} = \int_0^1 \frac{dx}{1+x^a},$$

for each positive integer a. The cases $a = 1$ and $a = 2$ give well-known and easy to find results. For $a > 2$, the technique of partial fractions makes it possible to evaluate the integrals. Some results on power series, such as term by term integration and, more importantly, Abel's Theorem (see Rudin [**3**]), are required to prove that the sum of the series and the value of the integral are the same.

4. The result of this paper cannot be extended to series $\sum_{i=1}^{\infty}(-1)^{i+1}/(ai+b)^u$, where u is a positive real number. If $u > 1$, then the series converges absolutely and all rearrangements have the same sum. If $0 < u < 1$, then all rearrangements of the type considered here diverge. (It is not difficult to show that the sequence $\{r_{(p+q)n}\}$ is unbounded for these values of u.)
5. A version of this argument for the alternating harmonic series ($a = 1$ and $b = 0$) can be found in Klambauer [**2**]. The paper [**1**] also considers the alternating harmonic series; the authors look at more general rearrangements than those discussed here. The results and methods of the present paper and [**1**] yield the following result. Let $\sum_{i=1}^{\infty} c_i$ be any rearrangement of the series $\sum_{i=1}^{\infty}(-1)^{i+1}/(ai+b)$ for which the positive terms remain in their original order as do the negative terms. (In other words, the subsequence of $\{c_i\}$ that consists of all the positive terms is decreasing to 0 and the subsequence of $\{c_i\}$ that consists of all the negative terms is increasing to 0.) For each positive integer n, let p_n be the number of positive terms in the set $\{c_1, c_2, \ldots, c_n\}$. Then the rearrangement $\sum_{i=1}^{\infty} c_i$ converges if and only if the sequence $\{p_n/n\}$ converges to a number in the interval (0, 1). If v is the limit of $\{p_n/n\}$, then the sum of the rearranged series is

$$S+\frac{1}{2a}\ln\left(\frac{v}{1-v}\right).$$

The details of the proof are a bit more tedious than those presented here, but the general approach should be clear after reading the two relevant papers.

6. For pedagogical purposes, the series

$$1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\cdots$$

illustrates the rearrangement theorem very well. The rearrangement of this series with p positive terms followed by q negative terms is easily shown to converge to $\ln(p/q)$.

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Generating Functions and the Electoral College

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What is the probability of a tie in the Electoral College? That is, if there are only two candidates, what is the chance that they each earn 269 of the 538 electoral votes? The solution given here involves generating functions. I think students in a combinatorics class would enjoy it.

We need some data. The following table indicates the number of electoral votes a state has, and the number of states that have that many votes. The values are those determined by the 2000 U.S. census.

Votes	3	4	5	6	7	8	9	10	11	12	13	15	17	20	21	27	31	34	55
States	8	5	5	3	4	2	3	4	4	1	1	3	1	1	2	1	1	1	1

This information can be stored in a generating function that will very elegantly calculate the number of ways any number of votes can be obtained. Let

$$\begin{aligned}p(x)&=(1+x^3)^8(1+x^4)^5(1+x^6)^3(1+x^7)^4(1+x^8)^2(1+x^9)^3(1+x^{10})^4(1+x^{11})^4\\&\quad\times(1+x^{12})(1+x^{13})(1+x^{15})^3(1+x^{17})(1+x^{20})(1+x^{21})^2(1+x^{27})\\&\quad\times(1+x^{31})(1+x^{34})(1+x^{55}).\end{aligned}$$

The coefficients of the various powers of x when this is multiplied out will give the number of ways to obtain a given number of votes. For example, using Mathematica, we find the coefficient of x^{269} is 17,057,441,245,652. The probability of getting exactly 269 votes is this number divided by $p(1)$, which is 0.00758. Of course, this is assuming that all possible combinations of votes are equally likely, which may not be realistic.