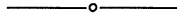


Students in beginning differential equations courses should recognize (4) as a standard Bernoulli-type first-order differential equation. Some of them might enjoy solving it for  $x = x(y)$  using techniques from their differential equations course, and then comparing that answer with its inverse  $y = y(x)$  obtained by integrating (3) via the substitution  $z = x^{-n}$ .



### Deriving the Equations of the Ellipse and Hyperbola

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Bill Bompert [TYCMJ 13 (1982) 198] describes a method for solving radical equations by using appropriate substitutions to transform the radical equation to a system of equations. The purpose of this note is to apply the same technique to the derivation of the equations of the ellipse and hyperbola.

Let  $F_1(-c, 0)$  and  $F_2(c, 0)$  be the foci of the ellipse centered at the origin, and let  $P(x, y)$  be any point on the ellipse. Then by the definition of an ellipse

$$d_1 + d_2 = 2a, \tag{1}$$

where  $d_1 = F_1P$  and  $d_2 = F_2P$ . Using the distance formula,

$$d_1^2 = (x + c)^2 + y^2 \quad \text{and} \quad d_2^2 = (x - c)^2 + y^2, \tag{2}$$

we obtain

$$d_1^2 - d_2^2 = 4cx. \tag{3}$$

From (1) and (3), it follows that

$$d_1 - d_2 = \frac{2cx}{a}. \tag{4}$$

Solving (1) and (4), we obtain

$$d_1 = a + \frac{c}{a}x \quad \text{and} \quad d_2 = a - \frac{c}{a}x \tag{5}$$

which are important in their own right since they give rational expressions for the distances from any point of the ellipse to its foci. Combining (2) and (5), we have

$$(x + c)^2 + y^2 = \left(a + \frac{c}{a}x\right)^2 \quad \text{and} \quad (x - c)^2 + y^2 = \left(a - \frac{c}{a}x\right)^2.$$

To obtain the equation of the ellipse, just simplify either equation after substituting  $b^2$  for  $a^2 - c^2$ . In a similar manner, the equation for the hyperbola can be derived.

Equations (5) provide some additional information. Observing that  $c/a = \epsilon$ , where  $\epsilon$  is the eccentricity of an ellipse, we write (5) as

$$d_1 = a + \epsilon x \quad \text{and} \quad d_2 = a - \epsilon x. \tag{6}$$

Recalling that the directrices are defined by the equations  $x = -a/\epsilon$  and  $x = a/\epsilon$ , we find the distances

$$D_1 = \frac{a}{\epsilon} + x \quad \text{and} \quad D_2 = \frac{a}{\epsilon} - x \quad (7)$$

from  $P(x, y)$  to the directrices. Thus, from (6) and (7),

$$\frac{d_i}{D_i} = \epsilon \quad (i = 1, 2). \quad (8)$$

Relations (8) take place for the ellipse, hyperbola and parabola; they may be considered as a unified definition of these loci. Finally, putting  $x = -c$  into the formula for  $d_1$  (or putting  $x = c$  into the formula for  $d_2$ ) in (5) yields the half-length of the latus rectum:

$$a - \frac{c^2}{a} = \frac{a^2 - c^2}{a} = \frac{b^2}{a}.$$

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