Students in beginning differential equations courses should recognize (4) as a standard Bernoulli-type first-order differential equation. Some of them might enjoy solving it for $x = x(y)$ using techniques from their differential equations course, and then comparing that answer with its inverse $y = y(x)$ obtained by integrating (3) via the substitution $z = x^{-n}$.

Deriving the Equations of the Ellipse and Hyperbola

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Bill Bompart [TYCMJ 13 (1982) 198] describes a method for solving radical equations by using appropriate substitutions to transform the radical equation to a system of equations. The purpose of this note is to apply the same technique to the derivation of the equations of the ellipse and hyperbola.

Let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci of the ellipse centered at the origin, and let $P(x, y)$ be any point on the ellipse. Then by the definition of an ellipse

$$d_1 + d_2 = 2a, \tag{1}$$

where $d_1 = F_1P$ and $d_2 = F_2P$. Using the distance formula,

$$d_1^2 = (x+c)^2 + y^2 \quad \text{and} \quad d_2^2 = (x-c)^2 + y^2, \tag{2}$$

we obtain

$$d_1^2 - d_2^2 = 4cx. \tag{3}$$

From (1) and (3), it follows that

$$d_1 - d_2 = \frac{2cx}{a}. \tag{4}$$

Solving (1) and (4), we obtain

$$d_1 = a + \frac{c}{a}x \quad \text{and} \quad d_2 = a - \frac{c}{a}x \tag{5}$$

which are important in their own right since they give rational expressions for the distances from any point of the ellipse to its foci. Combining (2) and (5), we have

$$(x+c)^2 + y^2 = \left(a + \frac{c}{a}x\right)^2 \quad \text{and} \quad (x-c)^2 + y^2 = \left(a - \frac{c}{a}x\right)^2.$$

To obtain the equation of the ellipse, just simplify either equation after substituting b^2 for $a^2 - c^2$. In a similar manner, the equation for the hyperbola can be derived.

Equations (5) provide some additional information. Observing that $c/a = \epsilon$, where ϵ is the eccentricity of an ellipse, we write (5) as

$$d_1 = a + \epsilon x \quad \text{and} \quad d_2 = a - \epsilon x. \tag{6}$$

Recalling that the directrices are defined by the equations $x = -a/\epsilon$ and $x = a/\epsilon$, we find the distances

$$D_1 = \frac{a}{\epsilon} + x \quad \text{and} \quad D_2 = \frac{a}{\epsilon} - x \tag{7}$$

from $P(x, y)$ to the directrices. Thus, from (6) and (7),

$$\frac{d_i}{D_i} = \epsilon \qquad (i = 1,2). \tag{8}$$

Relations (8) take place for the ellipse, hyperbola and parabola; they may be considered as a unified definition of these loci. Finally, putting $x = -c$ into the formula for d_1 (or putting $x = c$ into the formula for d_2) in (5) yields the half-length of the latus rectum:

$$a - \frac{c^2}{a} = \frac{a^2 - c^2}{a} = \frac{b^2}{a}.$$