

Two familiar trigonometric inequalities are immediate consequences of (1). If $f(x) = \sin x$ then $f''(x) = -\sin x < 0$ for $0 < x < \pi$. It follows that

$$\sum_1^n \sin x_i \leq n \sin \left(\frac{\sum_1^n x_i}{n} \right) \quad \text{for } 0 < x_i < \pi, \quad i = 1, 2, \dots, n,$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

If $f(x) = \cos x$ then $f''(x) = -\cos x < 0$ for $-\pi/2 < x < \pi/2$. Thus

$$\sum_1^n \cos x_i \leq n \cos \left(\frac{\sum_1^n x_i}{n} \right) \quad \text{for } -\frac{\pi}{2} < x_i < \frac{\pi}{2}, \quad i = 1, 2, \dots, n,$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Jensen's inequality also gives an immediate proof of the arithmetic-geometric mean inequality. Putting $f(x) = \ln x$, $f''(x) = -1/x^2 < 0$ for $x > 0$. Hence

$$\sum_1^n \ln x_i \leq n \ln \left(\frac{\sum_1^n x_i}{n} \right) \quad \text{for } x_i > 0, \quad i = 1, 2, \dots, n,$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Finally we note that if $f''(x) > 0$ then the inequality in (1) is reversed. Thus, for example, if $f(x) = \tan x$, $f''(x) = 2 \sec^2 x \tan x > 0$ for $0 < x < \pi/2$ and it follows that

$$\sum_1^n \tan x_i \geq n \tan \left(\frac{\sum_1^n x_i}{n} \right) \quad \text{for } 0 < x_i < \frac{\pi}{2}, \quad i = 1, 2, \dots, n$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

Pythagorean Theorem: $a \cdot a' + b \cdot b' = c \cdot c'$

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Proof. By similarity

$$\frac{x}{b} = \frac{b'}{a} \quad \text{or} \quad a \cdot x = b \cdot b' \quad \text{and} \quad \frac{y}{c} = \frac{c'}{a} \quad \text{or} \quad a \cdot y = c \cdot c'.$$

Therefore, $a \cdot a' = a \cdot (x + y) = b \cdot b' + c \cdot c'$.