$f(p_1,\ldots,p_n)=p_1^2/\lambda_1+\cdots+p_n^2/\lambda_n$ over the set of $(p_1,\ldots,p_n)$ satisfying the constraint $g(p_1,\ldots,p_n)=p_1+\cdots+p_n=p$, a given constant. The usual method of Lagrange multipliers applies here, and it is illuminating to view the solution in its equivalent form of "tangent level sets," expressing the fact that at a point $(p_1,\ldots,p_n)$ where an extremum is attained the gradient vector of f is parallel to the gradient vector of g. This gives once more the condition $p_i/\lambda_i=p_j/\lambda_j$ for all $i,j=1,\ldots,n$. Note that this point $(p_1,\ldots,p_n)$ is where the tangent hyperplane of the ellipsoid $p_1^2/\lambda_1+\cdots+p_n^2/\lambda_n=p^2/(\lambda_1+\cdots+\lambda_n)$ has equation $p_1+\cdots+p_n=p$, making equal angles with the coordinate axes.

As a final remark, we note that analogous problems in higher dimensions are easily solved by the same approach. For example, utilizing the convexity of the function $f(x)=x^{3/2}$, the reader may verify that n three-dimensional solids of prescribed shapes and respective surface areas $S_1,\ldots,S_n$, with $S_1+\cdots+S_n=S$, a given constant, have minimum total volume when

$$S_i=\frac{\lambda_i}{\lambda_1+\cdots+\lambda_n}S,\quad i=1,\ldots,n,$$

where λ_i is the analogue of the isoperimetric quotient, defined by the similarity invariant $\lambda_i=S_i^3/V_i^2$, for a solid with surface area S_i and volume V_i.

References

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A Productive Error in a Trigonometry Text

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Textbook misprints and ill-posed problems can be sources of frustration for students and authors alike. Sometimes, however, they may actually enhance learning by encouraging deeper exploration of concepts and discovery of relationships that might otherwise be overlooked. In this capsule we consider a trigonometry exercise which has appeared in several editions of a popular textbook series, despite the fact that it is ill-posed and the answer given in the text is not feasible. It is apparently intended as a routine exercise using the law of cosines, but we shall see that it can be used for other instructional purposes as well.

> *Two ships S and T are visible from an airplane A flying at an altitude of* 10,000 *feet* (*see Figure* 1). *If the angles of depression from A to S and T are* 37° *and* 21°, *respectively, and* $\angle SAT$ *is* 130°, *determine the distance between the ships.*

The author's intent seems clear. Using right triangle trigonometry on both $\triangle AWS$ and $\triangle AWT$ yields $\overline{AS}\approx 16{,}616$ feet and $\overline{AT}\approx 27{,}904$ feet, respectively. Applying the law of cosines to $\triangle SAT$ then gives $\overline{ST}\approx 40{,}630$ feet, the answer in the text (using four significant figures rather than five in the computations has little effect on the value obtained for $\overline{ST}$). Before we show that $\overline{ST}$ cannot possibly have this value, we mention two features of the exercise that seem to cause trouble for some students.

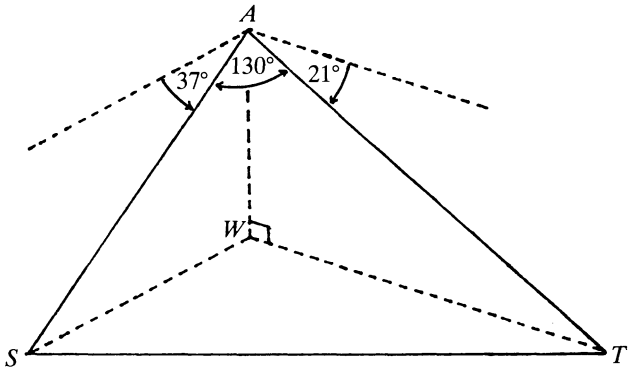

Figure 1

(1) Some have difficulty visualizing the three-dimensional nature of the figure and ask how the sum of the three angles can exceed 180°. The immediate response would seem to be that because these angles are in different planes, they are independent. However, as we shall see, this is incorrect!

(2) Some assume that $\angle SWT = \angle SAT = 130°$ and attempt to find $\overline{ST}$ by applying the law of cosines to $\triangle SWT$. Since $\overline{WS} \approx 13{,}270$ feet and $\overline{WT} \approx 26{,}051$ feet, using $\angle SWT = 130°$ leads to $\overline{ST} \approx 36{,}043$ feet, disagreeing with the textbook value for $\overline{ST}$. This suggests that a study of the relationship between $\angle SAT$ and its projection $\angle SWT$ might be in order.

An instructor can encourage students to determine the measure of $\angle SWT$ by applying the law of cosines to $\triangle SWT$ with $\overline{WS} \approx 13{,}270$ feet, $\overline{WT} \approx 26{,}051$ feet, and $\overline{ST} \approx 40{,}630$ feet. However, this leads to $\cos(\angle SWT) = -1.1514$, suggesting that perhaps the original exercise is ill-posed. This suggestion is reinforced by noting that the three lengths violate the triangle inequality! Since the angles of depression must surely be independent, the given measure of $\angle SAT$ must be considered highly suspect.

For purposes of analysis let us make no direct assumption about the measure of $\angle SAT$. Rather, let us observe that the *extreme* configurations are easily visualized. Indeed, the *maximal* values for both $\overline{ST}$ and the measure of $\angle SAT$ occur when S, W, and T are collinear as in Figure 2(a). Notice in this case that $\angle SAT$ is the supplement of the sum of the two angles of depression. Thus, when the given angles of depression are 37° and 21°, the largest possible measure for $\angle SAT$ is 122°, so the choice of 130° in the exercise does indeed create an inconsistency. Using the values of $\overline{AS}$ and $\overline{AT}$ mentioned earlier, we find that *maximal* $\overline{ST} \approx 39{,}321$ feet.

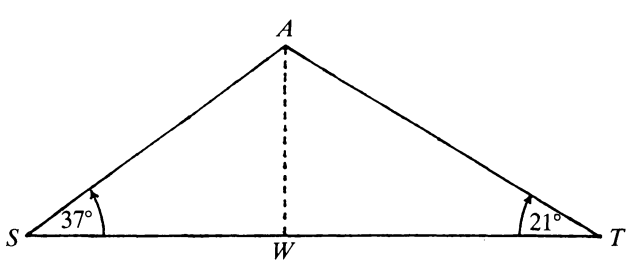

Figure 2(a)

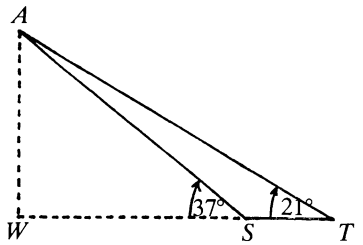

Figure 2(b)

Likewise, Figure 2(b) depicts the situation in which *minimal* $\overline{ST} \approx 12{,}781$ feet. Notice in this case that $\angle SAT = 37° - 21° = 16°$.

By generalizing the given exercise slightly, we can derive an interesting relationship between $\angle SAT$ and its projection $\angle SWT$. Refer to Figure 3. We assume $0° < \theta_2 \leq \theta_1 < 90°$ and imagine revolving $\triangle AWS$ around segment AW from the position for minimal $\overline{ST}$ (when $\theta_3 = \angle SWT = 0°$) to that for maximal $\overline{ST}$ (when $\theta_3 = 180°$). Correspondingly, $\theta_4 = \angle SAT$ varies over the interval

$$\theta_1 - \theta_2 \leq \theta_4 \leq 180° - (\theta_1 + \theta_2). \tag{1}$$

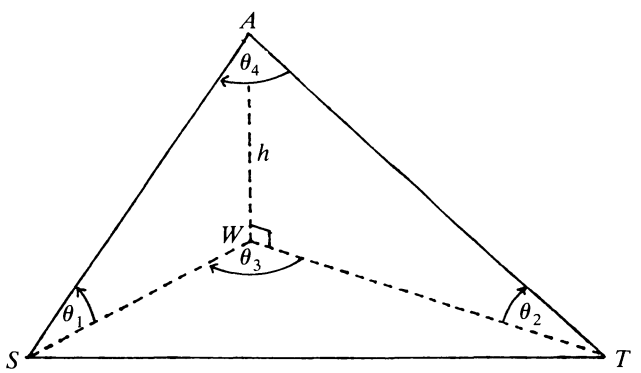

Figure 3

Applying the law of cosines to both $\triangle SAT$ and $\triangle SWT$ gives

$$\overline{ST}^2 = \overline{SA}^2 + \overline{AT}^2 - 2\overline{SA}\,\overline{AT}\cos\theta_4 = \overline{SW}^2 + \overline{WT}^2 - 2\overline{SW}\,\overline{WT}\cos\theta_3.$$

But $\overline{SA} = h\csc\theta_1$, $\overline{AT} = h\csc\theta_2$, $\overline{SW} = h\cot\theta_1$, and $\overline{WT} = h\cot\theta_2$. Thus,

$$\begin{aligned}
\cos\theta_4 &= \frac{\overline{SA}^2 + \overline{AT}^2 - \overline{SW}^2 - \overline{WT}^2 + 2\overline{SW}\,\overline{WT}\cos\theta_3}{2\overline{SA}\,\overline{AT}} \\
&= \frac{\csc^2\theta_1 + \csc^2\theta_2 - \cot^2\theta_1 - \cot^2\theta_2 + 2\cot\theta_1\cot\theta_2\cos\theta_3}{2\csc\theta_1\csc\theta_2} \\
&= \frac{2 + 2\cot\theta_1\cot\theta_2\cos\theta_3}{2\csc\theta_1\csc\theta_2} \\
&= \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\cos\theta_3. \qquad (2)
\end{aligned}$$

When θ_1 and θ_2 are fixed, we see from (2) that θ_4 varies directly with θ_3 and that $\cos\theta_4$ varies *linearly* with $\cos\theta_3$. Moreover, because of the identities

$$\begin{aligned}\sin\theta_1\sin\theta_2 &= [\cos(\theta_1-\theta_2)-\cos(\theta_1+\theta_2)]/2,\\ \cos\theta_1\cos\theta_2 &= [\cos(\theta_1-\theta_2)+\cos(\theta_1+\theta_2)]/2,\\ \cos(\theta_1+\theta_2) &= -\cos[180^\circ-(\theta_1+\theta_2)],\end{aligned}$$

we also see that the "coefficients" in (2) may be expressed in terms of the maximal and minimal measures of θ_4. Using these identities and letting $\theta_3 = 0^\circ$ and $\theta_3 = 180^\circ$ in (2) confirms the results already noted in (1). Also noteworthy is the special case when $\theta_3 = 90^\circ$, which yields $\cos\theta_4 = \{\cos(\theta_1-\theta_2)+\cos[180^\circ-(\theta_1+\theta_2)]\}/2$. Thus, when θ_3 achieves the mean of its extreme values, $\cos\theta_4$ is the mean of the cosines of the extreme values of θ_4.

Finally, since θ_4 increases from $\theta_1-\theta_2$ to $180^\circ-(\theta_1+\theta_2)$ as θ_3 increases from 0° to 180°, there must be a particular measure at which θ_4 and its projection θ_3 are equal. Letting $\theta_4 = \theta_3 = \theta$ in (2), we find that

$$\begin{aligned}\theta &= \arccos\left[\frac{\sin\theta_1\sin\theta_2}{1-\cos\theta_1\cos\theta_2}\right]\\ &= \arccos\left[\frac{\{\cos(\theta_1-\theta_2)+\cos[180^\circ-(\theta_1+\theta_2)]\}/2}{1-\{\cos(\theta_1-\theta_2)-\cos[180^\circ-(\theta_1+\theta_2)]\}/2}\right].\end{aligned}$$

Thus, the value at which $\theta_3 = \theta_4$ is always less than 90° and may be interpreted as a function of the maximal and minimal measures of θ_4. For the given exercise with $\theta_1 = 37^\circ$ and $\theta_2 = 21^\circ$, we obtain $\theta \approx 32^\circ$. Although these three angles sum to approximately 90°, such is not the case for $\theta_1+\theta_2+\theta$ in general.

The Volume and Centroid of the Step Pyramid of Zoser

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The Step Pyramid of Pharaoh Zoser at Saqqara, the most ancient of the Egyptian pyramids, was built around 2780 B.C. by the King's minister Imhotep, the official in charge of constructing a suitable sepulcher for his lord. The passage of five millennia has, of course, made Zoser's edifice a less perfect example of the ideal step pyramid than it originally was. We may also note that as the centuries progressed, the Egyptians built step pyramids with a greater number of steps, although these steps were of smaller altitude; by the time the monuments at Giza were put up, they had learned to add an external limestone incrustation to cover up the steps and make the four faces planar. However, visitors to the three pyramids of Giza will observe that this excrescence has been removed or has fallen away with time, except from the top of Chephren's structure; they will also see that the stories there are boxes with square bases.

One of the great achievements of Egyptian mathematics was the discovery of the formula for the volume of the frustum of a pyramid with square bases of length b and B and height h [7, Vol. 2, pp. 2–12]:

$$V = \frac{h}{3}(b^2+bB+B^2).$$