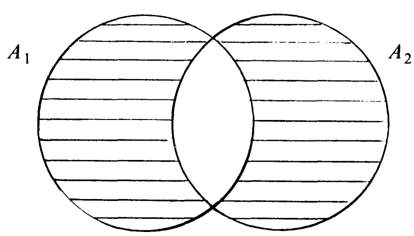
An Odd Induction Proof

Karl David, Middlebury College, Middlebury, VT

In a recent set theory class, students were asked to sketch a Venn diagram for the symmetric difference of three sets. One student tried to extend this to four sets; not surprisingly, he missed some cases. However, in perusing his attempt, I detected a pattern whose proof supplies a pretty and fresh example of mathematical induction, and which produces as a corollary a tidy proof of the fact that the symmetric difference operation is associative.

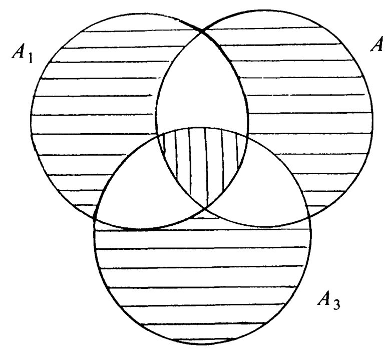
Recall (see the shaded region in Figure 1) that the *symmetric difference* of two sets A and B is

$$A \triangle B = (A - B) \cup (B - A).$$



$R_2 = A_1 \triangle A_2$

Figure 1.



$R_3 = (A_1 \triangle A_2) \triangle A_3$

Figure 2.

For any (not necessarily distinct) collection of sets $\{A_1, A_2, \ldots\}$, let $R_2 = A_1 \triangle A_2$ and define $R_k = R_{k-1} \triangle A_k$ for each $k > 2$ (see, for example, Figure 2). Then:

$$x \in R_n \text{ if and only if } x \text{ belongs to an odd number of sets in the collection } \{A_1, A_2, \ldots, A_n\}. \qquad (*)$$

The case $n = 2$ is clear by definition of $A_1 \triangle A_2$. Now, assuming that $(*)$ is true for $n \in \{3, 4, \ldots, N-1\}$, we shall show that it holds for $n = N$.

Suppose first that

$$x \in R_N = (R_{N-1} - A_N) \cup (A_n - R_{N-1}).$$

Then either $x \in R_{N-1} - A_N$ (so that $(*)$ holds for $n = N - 1$, but $x \notin A_N$) or $x \in A_N - R_{N-1}$ (so that $x \in A_N$, but x belongs to an even number of sets in the collection $A_1, A_2, \ldots, A_{N-1}$). In either case, we see that $(*)$ holds now for $n = N$, as desired.

Conversely, suppose that x belongs to an odd number of sets in the collection $\{A_1, A_2, \ldots, A_N\}$. Then either $x \in A_N$ (and so $x \notin R_{N-1}$ by our inductive hypothesis) or $x \notin A_N$ (in which case $x \in R_{N-1}$ by our inductive hypothesis). Thus, $x \in R_{N-1} \triangle A_N = R_N$.

For the case of $n = 3$, it follows from $(*)$ that $(A_1 \triangle A_2) \triangle A_3 =$

$$\{x\colon x \text{ belongs to precisely one set in } \{A_1, A_2, A_3\} \text{ or } x \in A_1 \cap A_2 \cap A_3\}.$$

Since the same reasoning and conclusion hold for $A_1 \triangle (A_2 \triangle A_3)$, the sets $(A_1 \triangle A_2) \triangle A_3$ and $A_1 \triangle (A_2 \triangle A_3)$ are identical. Thus, we have also established that the symmetric difference operation is associative.