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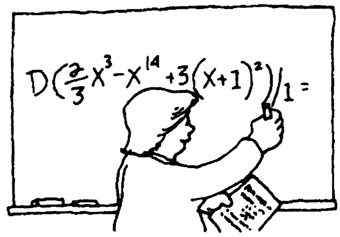


*Classroom Capsules* consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

## Proofs Without Words Under the Magic Curve

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There is no limit to the limits you can demonstrate under the magic curve $y = 1/x$. So you have dutifully defined the natural logarithm as an "area",

$$\ln t = \int_1^t \frac{1}{x}\,dx$$

for $t > 0$, and introduced $e$ as the positive number at which $\ln e = 1$. Before proving that $\ln ab = \ln a + \ln b$ by change of variables, please count to $\ln 485165195.4$ and read the following.

Would your students buy the fact that any two upper (or lower) "Riemann boxes" under the magic curve with equal endpoint ratios have the same area?

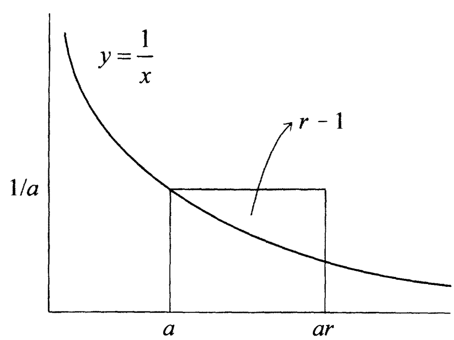


**Figure 1.**

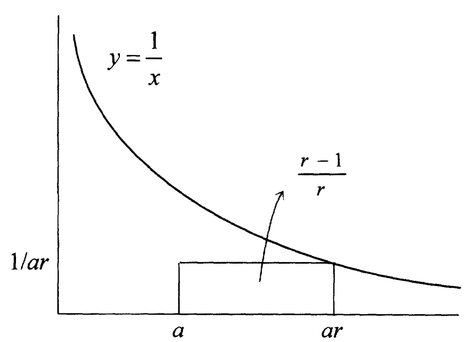


**Figure 2.**

Once they are hooked, show that this is true for actual areas under the curve.

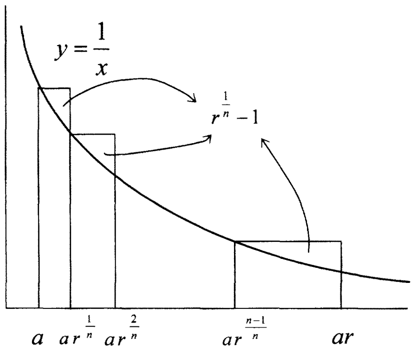


**Figure 3.**

The length of the longest subinterval in Figure 3, namely $\Delta x_n = ar^{\frac{n-1}{n}}(r^{1/n} - 1)$, goes to zero as $n$ goes to infinity! The limiting area is independent of $a$. This, of course, leads to

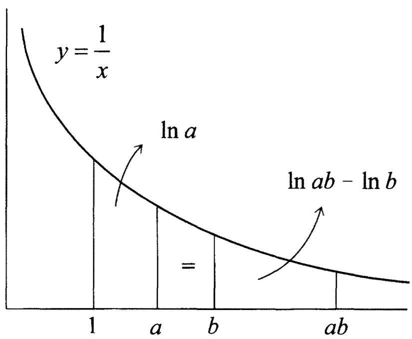


**Figure 4.**

$\ln ab = \ln a + \ln b$ because of Figure 4 (where $b \geq a > 1$, but the regions may overlap too). Then $\ln a = \ln(b \cdot (a/b))$ does the trick for the companion identity. As for the "down with the power!" rule, use the equal-area property again, with endpoints $1, a, a^2, \ldots, a^p$,

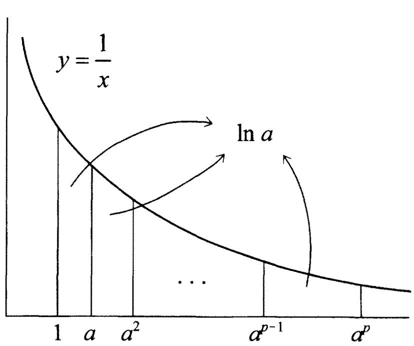


**Figure 5.**

to show that $\ln a^p = p \ln a$ ($p$ a positive integer) and use the fact that $q \ln(a^{1/q}) = \ln a$ ($q$ a positive integer) to conclude that $\ln(a^{p/q}) = (p/q) \ln a$ for a positive rational number $p/q$. In case of a negative power, change $a$ to $1/a$ and make the power positive.

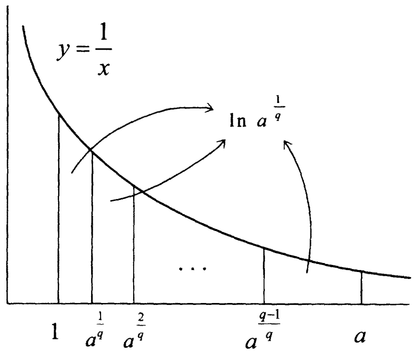


**Figure 6.**

Now comes the dreaded part where you prove that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e,$$

probably by using "logarithmic limits. That is your prerogative, of course, but why not continue with the theme of equal areas?

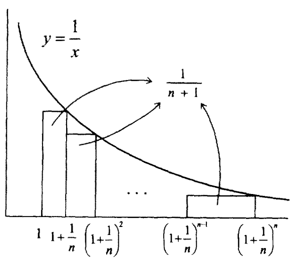


**Figure 7.**

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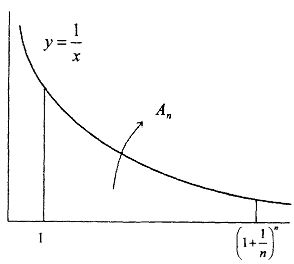


**Figure 8.**

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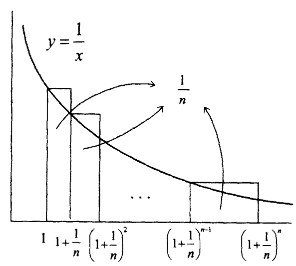


**Figure 9.**

It should not be too hard to convince our students that

$$\frac{n}{n+1} < A_n < 1 \quad \Rightarrow \quad A_n \to 1.$$

If the sandwiched-in area under $y = 1/x$ approaches 1, what do you think happens to $x = (1 + \frac{1}{n})^n$?

Why does the graph of the natural logarithm look the way it does? We can account for the $x$-intercept, the increase, and the negative concavity easily enough, but why do the ends go off to plus or minus infinity? Each of us has a little classroom strategy to deal with this problem, possible involving some inequalities, but please step under the magic curve once again and consider the following pictures:

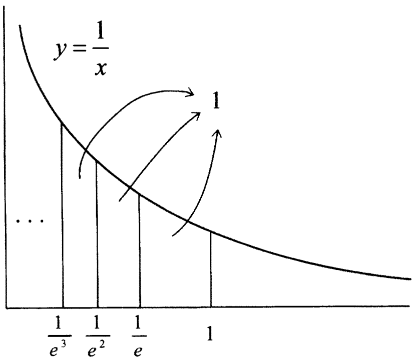


**Figure 10.**

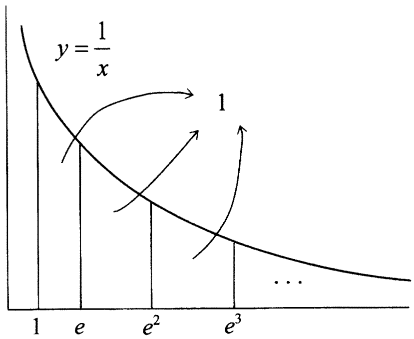


**Figure 11.**

Figure 10 shows that $\lim_{x\to 0^+} \ln x = -\infty$, and Figure 11 that $\lim_{x\to\infty} \ln x = \infty$.

Note that you can recycle these two pictures in Calculus II to "prove"

$$\int_0^1 \frac{1}{x}\,dx = \infty \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx = \infty.$$

Speaking of Calculus II, the telescoping series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1, \quad \text{with} \quad \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1},$$

also makes an appearance under the curve, though not through equal areas (see Figure 12).

Figure 13 (with $a > 1$) shows my all-time favorite bonus question on the subject of area.

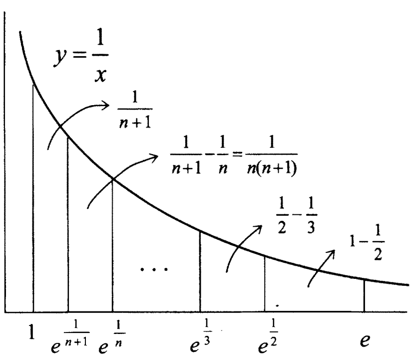


**Figure 12.**

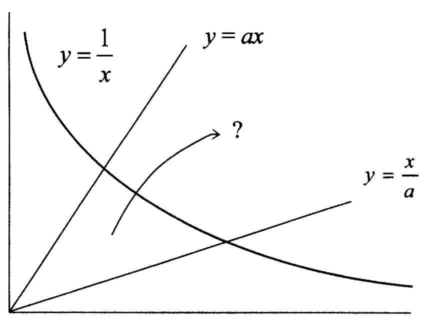


**Figure 13.**

The answer is $\ln a$, generalized from Figure 14 (with $a \geq b > 0$).

I think somebody (with tenure) ought to define the logarithm using Figure 13 and prove all properties of logarithms based on that, promising to ask for the proofs on the next test.

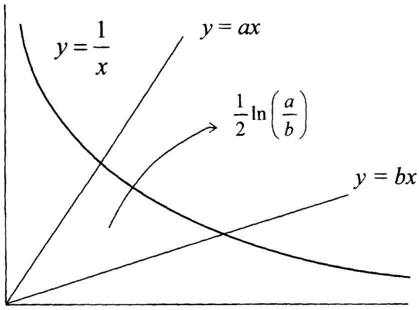


**Figure 14.**

*Acknowledgments.* I owe many thanks to Bogdan Mihaila and to Olcay Akman for technical assistance. I thank Prashant Sansgiry for pointing out two related, but different, Proofs Without Words that I include in the references. I don't claim to have thought of the proofs in this article before everybody else, but it certainly felt like that at the time.

## References

1. J. Ely, A visual proof that ln(ab) = ln a + ln b, *College Mathematics Journal* **27** (1996) 304.
2. A. H. Stein and D. McGavran, Proof of a common limit, *College Mathematics Journal* **29** (1998) 147.

## On A Mean Value Theorem

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Let $f$ be a function that is continuous on $[a, b]$ and differentiable on $(a, b)$. Denote by $M = (\frac{a+b}{2}, \frac{f(a)+f(b)}{2})$ the midpoint of the chord from $A = (a, f(a))$ to $B = (b, f(b))$, and let $P = (x, f(x))$ be any point on the graph. See Figure 1.

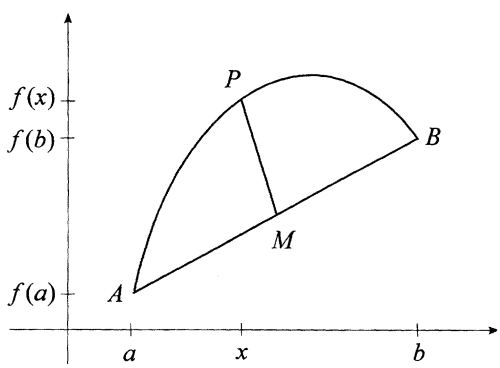


**Figure 1.**