type of incremental pricing policy in terms of feet, instead of yards, we need to determine $a$ and $d$ so that $S(a,d,600) = S(9,.03,200)$. Using (2), with $k = 3$, we have

$$a + 299.5d = 3.995.$$

Thus, there are many pricing solutions to our problem. Using (3), for example, we obtain $d = .03/9 = 1/300$ and $a = 3 - 1/300 = 2.99\bar{6}$. In particular, one solution would be to pay \$$2.99\bar{6}$ for drilling the first foot and 1/3 cent more for each additional foot.

The class can now see how practical mathematics may be. Indeed, from (2) and (3), we obtain

$$S\big(\alpha/k - (k-1)\delta/2k^2, \delta/k^2, km\big) = S(\alpha,\delta,m) \tag{4}$$

and

$$S(a,d,km) = S\big(ka + k(k-1)d/2, k^2d, m\big). \tag{5}$$

If your contractor offers to use units which are smaller by a factor of $1/k$ (e.g., feet instead of yards) so "you only will pay for what you use," then you must make sure that the price for the first unit is reduced to $\alpha/k - (k-1)\delta/2k^2$ and that the price for each additional unit is reduced by a factor of $1/k^2$. On the other hand, if your contractor insists on using units that are larger by a factor of $k$ "to reduce his overhead," while only increasing his pricing increment by the same factor, by all means do it. The savings $m^2k(k-1)d/2$ may more than pay for your subscription to this journal.

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## More Applications of the Mean Value Theorem

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The Mean Value Theorem applied to $f(x) = \log x$ on $[a,b]$ yields $\log b - \log a = (b-a)(1/c)$ for some $c \in (a,b)$. This can be recast as

$$(b-a)/b < \log b/a < (b-a)/a \tag{1}$$

in order to obtain efficient proofs of the following:

(i) $n^m > m^n$ if $m > n \geqslant e$ and $n^m < m^n$ if $e \geqslant m > n > 0$.

(ii) $\sqrt[n]{a_1a_2\cdots a_n} \leqslant (1/n)\sum_{i=1}^{n} a_i$ ($a_i$ positive) with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

(iii) $(1+1/n)^n < e < (1+1/n)^{n+1}$ for all positive integers $n$.

To establish (i), let $a = n$ and $b = m$. If $n \geqslant e$, the the right-hand inequality in (1) yields

$$(m/n)^n < e^{m-n} \leqslant n^{m-n},$$

from which $m^n < n^m$ follows. If $e \geqslant m$ and $n > 0$, the left-hand inequality in (1) yields

$$m^{(m-n)/m} \leqslant e^{(m-n)/m} < m/n.$$

Therefore, $m^{m-n} < (m/n)^m$ and $m^{n-m} > (n/m)^m$, so that $m^n > n^m$. It is of interest to note that a special case of the first part of (i) is the familiar result $e^{\pi} > \pi^e$.

To establish (ii), let $a = 1$. Then (1) becomes

$$(b-1)/b < \log b < b - 1. \tag{2}$$

Using the left-hand inequality in (2), we get $1 - 1/b < \log b$, or $1/b - 1 > -\log b = \log 1/b$. Since $b > 1$, it follows that $1/b \in (0,1)$. Combining this with the right-hand inequality in (2), we have shown that for any $x > 0$:

$$x - 1 \geqslant \log x, \text{ with equality holding if and only if } x = 1. \tag{3}$$

Now let

$$a = (1/n)\sum_{i=1}^{n} a_i \quad \text{and} \quad g = \sqrt[n]{a_1 a_2 \cdots a_n}\,.$$

Applying (3) to each $a_i/g$ and adding, we obtain

$$(1/g)\left(\sum_{i=1}^{n} a_i\right) - n \geqslant \log\left[(a_1 a_2 \cdots a_n)/g^n\right],$$

or

$$(na)/g - n \geqslant 0.$$

Thus, $a \geqslant g$. Here, equality holds if and only if each $x = a_i/g$ in (3) equals 1; that is, if and only if $a_1 = a_2 = \cdots = a_n$.

Finally, to establish (iii), put $b = (n+1)/n$ in (2) to obtain $1/(n+1) < \log(n+1)/n < 1/n$. Thus, $\log(1 + 1/n)^n < 1 < \log(1 + 1/n)^{n+1}$, which gives (iii).

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**Cantor's Disappearing Table**

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It is useful occasionally to remind students that mathematics is more than formulas and rote procedures, and that even simple ideas may have logical consequences which confound intuition. The following graphic demonstration is based on properties of a "fat" Cantor set, and the only background required is limits of sequences and the sum of a geometric series.

To begin, choose a table (or any object) having finite length $l$. Challenge students to make the table disappear by removing exactly half of it. Intuition (freshman intuition) says this is impossible. To show that it is not, mark the midpoint of the table and remove the section of length $l/4$ centered at the midpoint. (A piece of chalk applied to the edge of the table graphically shows the part of the table which is to be removed.) At this point, $1/4$ of the table has been removed and each of the remaining pieces has length less than $l/2$. Next, remove $1/8$ of the table by taking sections of length $l/16$ from the centers of the two remaining pieces. The amount of the table which has been removed is now $(1/4) + (1/8)$, and each of the remaining pieces has length less than $l/4$. Continue the process. The result is a table marked as shown, and two strings of numbers: