s	True $P(S = s)$	Normal PDF Approximation
7	0.0015	0.0017
8	0.0056	0.0059
9	0.0164	0.0165
10	0.0389	0.0384
11	0.0752	0.0742
12	0.1197	0.1188
13	0.1576	0.1575
14	0.1721	0.1730
15	0.1563	0.1575
16	0.1182	0.1188
17	0.0744	0.0742
18	0.0388	0.0384
19	0.0168	0.0165
20	0.0060	0.0059
21	0.0017	0.0017

Classroom Experience. In the fall of 1997, the normal table and the normal pdf approximations were demonstrated to students in a calculus-based probability class. The final exam for the class included the following problem: Let $S = X_1 + \cdots + X_{150}$, where the X's are independent, $P(X_i = -1) = P(X_i = 0) = P(X_i = 1) = 1/3$. Find a normal approximation to $P(S = 5)$. All eight students gave correct solutions. Two used a normal table approximation. Five chose a normal pdf approximation. The remaining student used a pdf approximation, then checked his answer with a normal table calculation.

Overall, the introduction of the normal pdf approximation method seemed appropriate for the class and enriching for the students. Its introduction will be included the next time the class is taught.

References

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Normal Lines and Curvature

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Let C be a differentiable curve whose equation is $y = f(x)$. It is standard practice in calculus textbooks to introduce the tangent line to C at the point $P(a, f(a))$ by considering a nearby point $Q(a + h, f(a + h))$, where $h \neq 0$, then considering the slope m_{PQ} of the secant line PQ and defining the tangent line to C at P to be the line containing $P(a, f(a))$ and having slope $m = \lim_{h \to 0} m_{PQ}$.

In the spirit of the above let us replace the tangent line to C at $P(a, f(a))$ by the normal line

$$N_P : y - f(a) = -\frac{1}{f'(a)}(x - a) \qquad (f'(a) \neq 0)$$

and replace the secant line PQ by the normal line at Q,

$$N_Q : y - f(a+h) = -\frac{1}{f'(a+h)}(x - (a+h)) \qquad (f'(a+h) \neq 0).$$

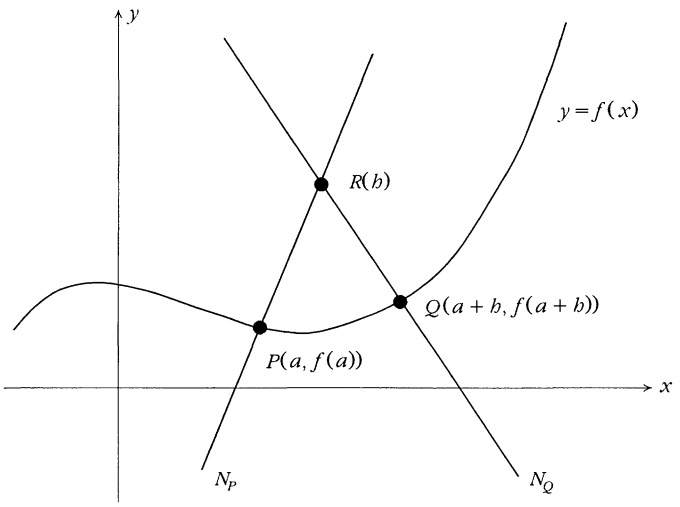

Figure 1

We rewrite the equations so as to allow vertical normal lines:

$$\begin{aligned} N_P &: x + f'(a)y = a + f(a)f'(a), \\ N_Q &: x + f'(a+h)y = a + h + f(a+h)f'(a+h). \end{aligned} \tag{1}$$

We assume that C is a twice differentiable curve and that the zeroes of f'' are isolated. Thus there is an interval I about a such that $f'(a+h) \neq f'(a)$ for all $a + h \neq a$ in I. So for all such points in I, the normal lines N_P and N_Q are not parallel. Let $R(h) = (x_h, y_h)$ denote their point of intersection. What happens to this intersection point as $h \to 0$?

To find the limiting point $(x_0, y_0) = R = \lim_{h \to 0} R(h) = (\lim_{h \to 0} x_h, \lim_{h \to 0} y_h)$ we solve (1), by Cramer's rule or otherwise, to find the point $R(h)$. We have

$$x_h = \frac{a(f'(a+h) - f'(a)) - (f(a+h) - f(a))f'(a)f'(a+h) - hf'(a)}{f'(a+h) - f'(a)}$$

and

$$y_h = \frac{h + f(a+h)f'(a+h) - f(a)f'(a+h) + f(a)f'(a+h) - f(a)f'(a)}{f'(a+h) - f'(a)}$$

Divide the numerator and the denominator of the fractions by h and let $h \to 0$ to obtain (if $f''(a) \neq 0$)

$$x_0 = \lim_{h \to 0} x_h = \frac{af''(a) - f'(a)^3 - f'(a)}{f''(a)} = a - \frac{f'(a)^3}{f''(a)} - \frac{f'(a)}{f''(a)}$$

and

$$y_0 = \lim_{h \to 0} y_h = \frac{1 + f'(a)^2 + f(a)f''(a)}{f''(a)} = f(a) + \frac{f'(a)^2}{f''(a)} + \frac{1}{f''(a)}.$$

So if $P(a, f(a))$ is such that $f''(a) \neq 0$ then

$$R = \left(a - \frac{f'(a)^3}{f''(a)} - \frac{f'(a)}{f''(a)}, f(a) + \frac{f'(a)^2}{f''(a)} + \frac{1}{f''(a)} \right).$$

The distance from R to the point $P(a, f(a))$ is

$$\frac{\left(1 + f'(a)^2\right)^{3/2}}{|f''(a)|},$$

which is the radius of curvature of C at $P(a, f(a))$. Moreover R is the center of the circle of curvature of C at $P(a, f(a))$, that circle that best approximates C at $P(a, f(a))$.

The above is undoubtedly known to differential geometers, but it is surprising that it has not made its way into calculus textbooks. Using a specific curve and a specific fixed point on the curve, it would be an instructive exercise to have students compute R by a limiting process. Indeed, when experienced calculus teachers were asked what happens to the point $R(h)$ as $h \to 0$, many felt that this intersection point would go to infinity.

This is also a nice introduction to the concept of curvature of a planar curve at a point. There are two extremes, a line and a circle. For a line, all normals are parallel and R is "infinity". For a circle, all normals pass through the center of the circle and R is the center of the circle. For a general curve $y = f(x)$, the analysis above leads one to observe that if $f''(a) \neq 0$, the curve acts locally like a circle because R is finite. If $f''(a) = 0$, then the curve acts locally like a line.

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A Picture for Real Arithmetic

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By means of a stereographic projection, one can map the real line to a circle and thus have a picture which stays on the page, instead of running off it. By means of geometric procedures, the product and sum of two numbers can then be constructed. Various properties of arithmetic, such as the product of two negatives being positive, can then be visually presented.

For a concrete example, consider an (x, y)-coordinate system, and map the x-axis to the unit circle with $(0, 1)$ as the projection point (Figure 1).

