*Exercise.* Suppose the stock is selling for \$90 a share and the option sells for \$10. Explain how a trader could make a \$5 profit with no risk.


### References

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## Candies and Dollars

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Several years ago, my brother Ali Adnan (may God rest his soul) gave me this problem.

> John was a young boy with a jar of candies which he consumed in the following manner. On the first day, he ate one candy and gave away exactly 10% of the remaining number. On the second day, he ate two candies and gave away exactly 10% of the remaining number. He followed this pattern each day until the jar was empty. How many candies did the jar originally contain?

This problem allows for a trivial solution: the jar could have held just one candy at the start. A nontrivial solution can easily be obtained by making a guess and working backwards. Suppose we guess that, after eating candy on the next-to-last day, John gave away 10% of 10 candies; then he ate nine candies on the last day, the ninth, by definition. That is, there were 18 candies when the eighth day began, 27 candies when the seventh day began, and so on. We conclude that the jar originally contained 81 candies. This method, however, sheds no light on the surprising uniqueness of this nontrivial solution. The following argument will derive a nontrivial solution to a generalized form of this problem and also prove the uniqueness of the solution.

Let $J_0$ be the original number of candies (assumed to be a positive integer) and let $J_r$ represent the number of candies in the jar at the end of the $r$th day. Thus, $\{J_r\}$ is to be a sequence of nonnegative integers with the property that $J_r$ is obtained from $J_{r-1}$ by subtracting $r$ and then subtracting $s\%$ of what remains. In general, $s$ does not have to be 10 (and does not have to be an integer), but we restrict it so that $n = 100/s$ is an integer. We claim that, in the unique nontrivial solution, the candies are finished off on day $n-1$ and the jar originally contained $(n-1)^2$ candies. To see this, we must analyze the relation $J_r = (J_{r-1} - r)[1 - (s/100)]$.

Setting

$$q = \left(1 - \frac{s}{100}\right)^{-1} = \left(1 - \frac{1}{n}\right)^{-1} = \frac{n}{n-1},$$

we get $J_r = (J_{r-1} - r)q^{-1}$. Thus,

$$J_{r-1} = qJ_r + r = q(qJ_{r+1} + r + 1) + r = q^2 J_{r+1} + r + (r+1)q,$$

and continuing in this fashion,

$$J_{r-1} = q^{k+1} J_{r+k} + \sum_{i=0}^{k} (r+i) q^i$$

for values of $k$ from 0 up to where $J_{r+k} = 0$. Setting $r = 1$ and assuming $k$ can be chosen so that $J_{k+1} = 0$ (which makes $k+1$ the day that the last candies are eaten), we obtain

$$J_0 = q^{k+1} J_{k+1} + \sum_{i=0}^{k} (1+i) q^i = 0 + \frac{d}{dq} \sum_{i=0}^{k} q^{i+1} = \frac{d}{dq}\left(\frac{q^{k+2}-q}{q-1}\right).$$

It follows that the original number of candies must be given by

$$J_0 = \frac{(k+1)q^{k+2} - (k+2)q^{k+1} + 1}{(q-1)^2}. \qquad (1)$$

If $k = 0$ (so that $J_1 = 0$), this gives the trivial solution to the problem: $J_0 = (q^2 - 2q + 1)/(q-1)^2 = 1$. On the other hand, if $k > 0$ (and $J_{k+1} = 0$), then we must consider how equation (1) can yield an integer value for $J_0$. Note that $q = n/(n-1)$ implies $q - 1 = 1/(n-1)$, so (1) becomes (after algebraic simplification)

$$J_0 = \left(\frac{n}{n-1}\right)^k n(k+2-n) + (n-1)^2. \qquad (2)$$

Since $n$ and $n-1$ are relatively prime, the right-hand side of (2) is an integer only when $n = k+2$. Consequently, day $k+1$ when the candy is finished is day $n-1$, and the original number of candies is $(n-1)^2$, as claimed. One can check that this value for $J_0$ does indeed lead to an integer sequence $\{J_r\}$. In fact, it is easily proved by induction that $J_r = (n-1)(n-r-1)$ for $r = 0, \ldots, n-1$.

As an exercise, here is a related problem that can be solved by similar means:

> On the $r$th day, a pool of $I$ dollars is increased by $r^2$ and then reduced by $s\%$ (where $n = 100/s$ is an integer). Assume that the pool has an integer number of dollars and that on some future day the original amount $I$ is restored. Show that $I = (n-1)^2(2n-1)$ is the unique nontrivial solution under these conditions and that, in this case, $2(n-1)$ is the number of days required.

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## A Simple Solution of the Cubic

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The quadratic formula for the general degree-two equation is one of the most familiar equations in mathematics. Surely every college mathematics teacher can quote it and derive it without effort. In contrast, the corresponding equation for the solution of the general cubic is quite obscure. We are all aware that such a formula exists, but it is an uncommon mathematician who can quote the result, let alone derive it from first principles. Imagine our surprise, therefore, at discovering a simple algebraic