The Spider's Spacewalk Derivation of sin′ and cos′

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The usual proofs of the derivatives of sine and cosine in introductory calculus involve limits. I shall outline a simple geometric derivation that avoids evaluating limits, based on the interpretation of the derivative as the instantaneous rate of change. The principle behind this proof is found in a late nineteenth-century calculus textbook by J. M. Rice and W. W. Johnson, *The Elements of the Differential Calculus, Founded on the Method of Rates or Fluxions* (Wiley, New York, 1874).

A spider walks with speed 1 in a circular path around the outside of a round satellite of radius 1, as shown in Figure 1. At time t the spider will have travelled a distance t, which corresponds to a central angle of t radians. The altitude of the spider, in the standard coordinate system, is $y = \sin(t)$ and the spider is $x = \cos(t)$ units to the right (or left) of the origin.

Now look how fast the spider is moving *upward*. Since the altitude of the spider at time t is $y = \sin(t)$, its upward velocity is $y' = \sin'(t)$. Oops!—The spider loses its footing at time t, and since the gravity in outer space is negligible, it continues with the same direction and speed. It moves a distance 1 in additional time $\Delta t = 1$,

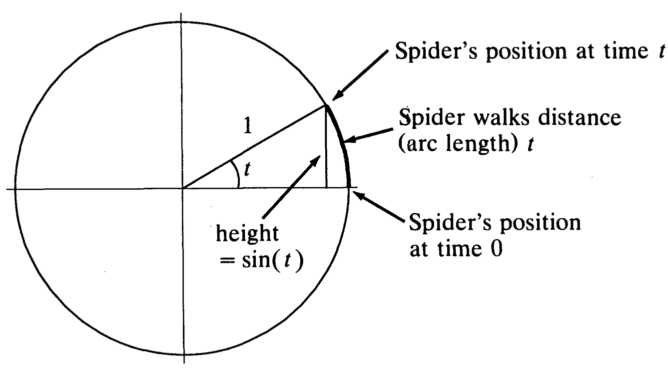

Figure 1

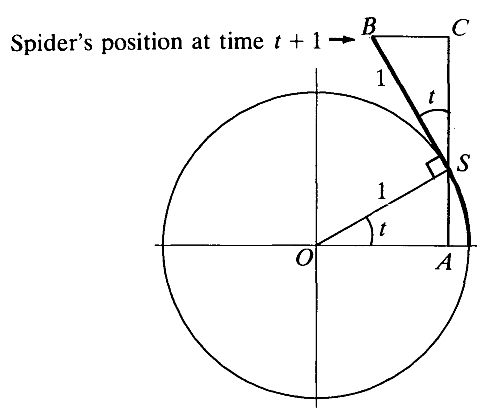

Figure 2

from point S to point B; see Figure 2. The spider's altitude changes by $\Delta y = SC$, so its upward velocity is $y' = \Delta y/\Delta t = SC$. But because triangles OAS and SCB are congruent ($\angle OSA = 90° - t$, so $\angle CSB = t = \angle AOS$), $SC = OA = x = \cos(t)$. Therefore $\sin'(t) = y' = \cos(t)$.

Similarly, the spider is $x = \cos(t)$ units to the right of the center, and its horizontal velocity is $x' = \cos'(t) = \Delta x/\Delta t = -BC = -SA = -\sin(t)$. The minus sign arises because the spider is moving to the left when it is above the x-axis, i.e., $\Delta x/\Delta t$ is negative whenever $y = \sin(t)$ is positive.

Figure 2 describes the case $0 < t < \pi/2$, but it is easily modified to yield the same results when the spider is located anywhere on the unit circle.

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[*Editor's note.* This proof also appeared in C. S. Ogilvy, Derivatives of $\sin\theta$ and $\cos\theta$, *American Mathematical Monthly* 67 (1960) 673.]

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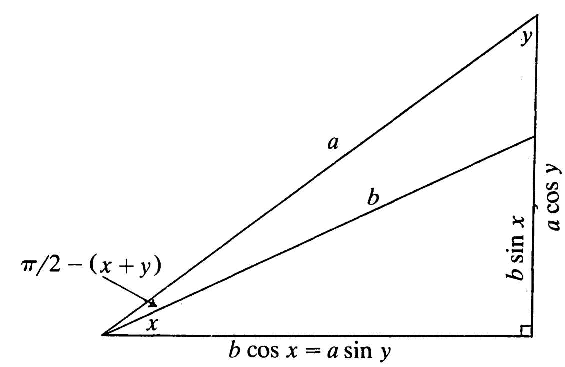
cos(x + y) for x + y > π/2



$$\tfrac{1}{2}ab\sin[\pi/2 - (x+y)] = \tfrac{1}{2}b\cos x \cdot a\cos y - \tfrac{1}{2}a\sin y \cdot b\sin x$$

$$\therefore \cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

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