$$bc\cos B = \frac{b^2 + c^2 - \overline{AC}^2}{2} = x_2^2 + y_2^2 - x_1x_2 - y_1y_2 - dx_2 + dx_1.$$

Summing these four equalities, with some minor rearrangements, gives

$$\begin{aligned} w &= ab\cos A + cd\cos C + ad\cos D + bc\cos B \\ &= \left[x_1^2 + 2dx_1 - 2x_1x_2 + d^2 - 2x_2d + x_2^2\right] + \left[y_1^2 + y_2^2 - 2y_1y_2\right] \\ &= 4\left[\left\{\frac{(x_1+d)}{2} - \frac{x_2}{2}\right\}^2 + \left\{\frac{y_1}{2} - \frac{y_2}{2}\right\}^2\right] \end{aligned}$$

and this last expression equals $4x^2$ since the midpoints have coordinates

$$M_1 = \left(\frac{x_1+d}{2}, \frac{y_1}{2}\right) \quad \text{and} \quad M_2 = \left(\frac{x_2}{2}, \frac{y_2}{2}\right).$$

We note in closing that if the vertices of the quadrilateral are located at $A : (0, 2)$, $B : (2n + 4, 2n + 2)$, $C : (4, 0)$, and $D : (0, 0)$, then the distance between the two midpoints is $x = n\sqrt{2}$, and hence can be made as large as desired, reflecting the fact that a quadrilateral can be quite unlike a parallelogram.

Acknowledgment. Inspiration for this article came from listening to Bill Dunham deliver the featured address on "Euler" at the Fall, 1999 meeting of the Ohio Section of the M.A.A.

References

1. Posamentier, Alfred S., *Excursions in Advanced Euclidean Geometry*, revised ed., Addison-Wesley, 1984, 96–106.
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The Volume of a Tetrahedron

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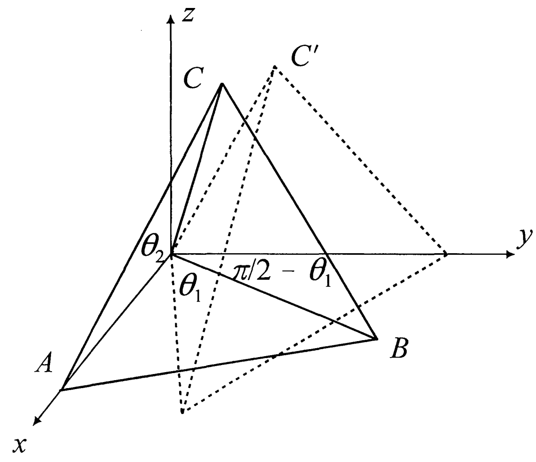
There are many formulas for calculating the area of a triangle, including one that is used when we know only the lengths of two sides and the angle between them. Likewise there are formulas for calculating the volume of a tetrahedron, but the common ones require the coordinates of the four vertices. Suppose, instead, we are given only the lengths of three edges having a common point and the measures of the three angles between these edges.

Theorem. For a tetrahedron $OABC$ let the angles $\angle AOB$, $\angle AOC$, and $\angle BOC$ have given values θ_1, θ_2, and θ_3, and let the lengths of the edges OA, OB, and OC be a, b, and c, respectively. Let $\theta = \frac{\theta_1+\theta_2+\theta_3}{2}$. Then the volume of the tetrahedron is given by

$$V = \frac{1}{3}abc\sqrt{\sin\theta\sin(\theta-\theta_1)\sin(\theta-\theta_2)\sin(\theta-\theta_3)}. \tag{2}$$



Proof. Put the tetrahedron in Euclidean 3-space as pictured. Since the area of $\triangle OAB$ is $\frac{1}{2}ab\sin\theta_1$, we only need to find the z-coordinate of the point C to calculate V. Let α, β, and γ be the angles that the vector $\overrightarrow{OC}$ makes with the positive x, y, and z axes (the direction angles). Then we know

$$\overrightarrow{OC} = c(\cos\alpha, \cos\beta, \cos\gamma)$$

and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{3}$$

Since the z-coordinate of C is $c \cdot \cos\gamma = c\sqrt{1 - \cos^2\alpha - \cos^2\beta}$ (using (3)) and we have chosen coordinates in such a way that $\alpha = \theta_2$, then the task that remains is to express $\cos\beta$ in terms of the given data. For this, as suggested in the picture, we rotate the point C about the z axis by the angle $\pi/2 - \theta_1$ to construct the new point C'. For vectors in the xy plane, the matrix of this rotation is $\begin{bmatrix} \sin\theta_1 & -\cos\theta_1 \\ \cos\theta_1 & \sin\theta_1 \end{bmatrix}$ so the y-coordinate of C' is $c(\cos\theta_1\cos\theta_2 + \sin\theta_1\cos\beta)$. By our choice of rotation, however, the angle between OC' and the positive y axis is θ_3, so the y-coordinate of C' is $c \cdot \cos\theta_3$. Equating these two expressions leads to

$$\cos\beta = \frac{\cos\theta_3 - \cos\theta_1\cos\theta_2}{\sin\theta_1}. \tag{4}$$

We are now able to use (3) and (4) to express $\cos\gamma$ in terms of the given data. In fact,

$$\begin{aligned}
\cos^2\gamma &= 1 - \cos^2\alpha - \cos^2\beta \\
&= \sin^2\theta_2 - \left(\frac{\cos\theta_3 - \cos\theta_1\cos\theta_2}{\sin\theta_1}\right)^2 \\
&= \frac{(\sin\theta_2\sin\theta_1 + \cos\theta_3 - \cos\theta_1\cos\theta_2)(\sin\theta_2\sin\theta_1 - \cos\theta_3 + \cos\theta_1\cos\theta_2)}{\sin^2\theta_1} \\
&= -\frac{(\cos(\theta_1+\theta_2) - \cos\theta_3)(\cos(\theta_1-\theta_2) - \cos\theta_3)}{\sin^2\theta_1}.
\end{aligned} \tag{5}$$

To simplify further we use a trigonometric identity that can be found in various sources:

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right). \tag{6}$$

Combining (5) and (6) then yields

$$\cos \gamma = \frac{2\sqrt{\sin\theta \sin(\theta - \theta_1) \sin(\theta - \theta_2) \sin(\theta - \theta_3)}}{\sin\theta_1}$$

(since $0 < \theta_1 < \pi$, $\sin\theta_1$ is positive).

Finally, the volume of the tetrahedron is given by

$$\begin{aligned} V &= \frac{1}{3}\left(\frac{1}{2}ab\sin\theta_1\right)(c \cdot \cos\gamma) \\ &= \frac{1}{6}abc\sin\theta_1 \frac{2\sqrt{\sin\theta \sin(\theta - \theta_1) \sin(\theta - \theta_2) \sin(\theta - \theta_3)}}{\sin\theta_1} \\ &= \frac{1}{3}abc\sqrt{\sin\theta \sin(\theta - \theta_1) \sin(\theta - \theta_2) \sin(\theta - \theta_3)}. \end{aligned}$$

Ten into Eight Won't Go?

Marc Brodie (College of St. Benedict, mbrodie@csbsju.edu), who does not get all his news from television, found an item showing that the Minneapolis *Star-Tribune* has an insufficient appreciation of the pigeonhole principle:

WRESTLING from C1

Ten Gophers in top eight; Iowa holds narrow lead