Using Seifert's theorem, which I stated in class but did not prove, and Stokes' theorem, we now know that

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 0$$

for any smooth simple closed curve C.

Incidentally, George Stokes got the idea for his theorem from an 1850 letter from Lord Kelvin **[2]**, the founder of knot theory **[1]**.

References

1. Michael Atiyah, *The Geometry and Physics of Knots*, Cambridge University Press, 1990, section 1.3.
2. C. H. Edwards and D. E. Penney, *Calculus with Analytic Geometry*, 4th ed., Prentice Hall, Englewood Cliffs, NJ, 1994, p. 948.
3. Dale Rolfsen, *Knots and Links*, Publish or Perish, Houston, TX, 1976.

Exploring Fibonacci Numbers Mod *M*

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Fibonacci numbers, commonly defined as the sequence $\{f(n)\}$ with $f(1)=1, f(2)=1$, and $f(n)=f(n-1)+f(n-2)$ for $n>2$, are used to teach the concept of recursion in mathematics and computer science courses. These numbers become large quickly, but by computing the Fibonacci sequence modulo a natural number m, we are able to keep the numbers being generated inside the interval $[0, m-1]$. Thus we fix m and consider the generalized Fibonacci sequence mod m defined by

$$\begin{aligned} f(1) &= a \\ f(2) &= b \\ f(n) &= f(n-1) + f(n-2) \bmod m \qquad \text{for } n>2 \end{aligned} \tag{1}$$

where the inputs a and b are chosen from $0, 1, \ldots, m-1$. For example, if $m=10$ and $a=b=1$, then the sequence begins

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, \ldots.$$

Is this sequence eventually periodic? If so, what is the cycle length? Must such a sequence cycle back to the initial value? Do all the numbers $0, \ldots, m-1$ appear? Do they appear equally often? How do the answers to these questions depend on m, a, and b? I was originally drawn to these questions when searching for an efficient (pseudo)random number generator.

A natural way to visualize the sequence $\{f(n)\}$ is to plot its graph, the sequence of points $(n, f(n))$, as in Figure 1. But some features will be more apparent in a plot of the *Fibonacci walk*, the sequence of points $P_n = (f(n), f(n+1))$. This is the

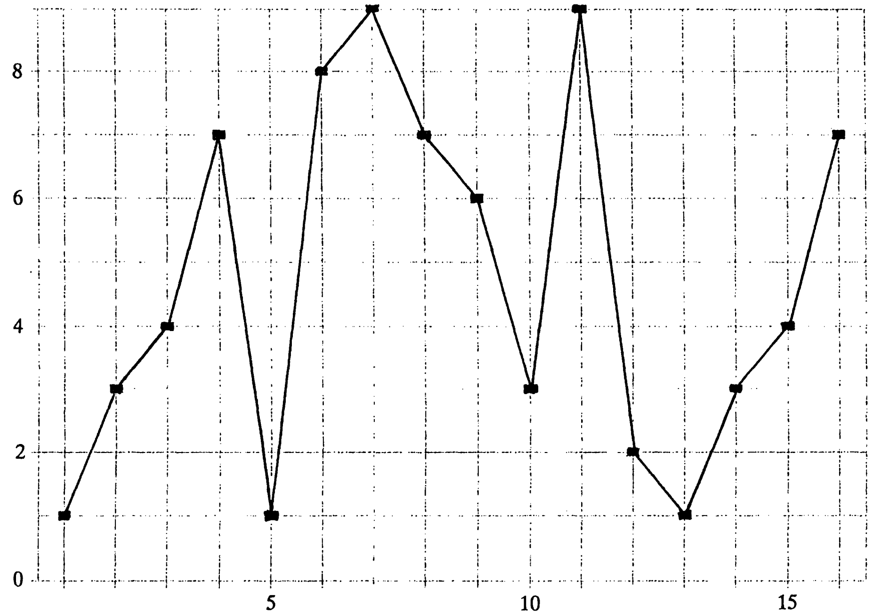

Figure 1. The graph of f using the data $m = 10, a = 1, b = 3$.

analogue for the discrete second-order difference equation (1) of the *phase-plane* plot of a solution to a second-order differential equation. The sequence $\{f(n)\}$ is eventually periodic if and only if the Fibonacci walk revisits one of its points, because the ordered pair $(f(n), f(n+1))$ uniquely determines the rest of the Fibonacci sequence. Of course, since the Fibonacci walk takes place on an $m \times m$ grid, points *must* be revisited and, in fact, cycle lengths have an upper bound of m^2. Figure 2 illustrates the Fibonacci walk for $m = 10$, $a = 1$, and $b = 3$, a cycle of length 12.

The Fibonacci sequence mod m has been studied, and some of the questions I raised have been solved in the literature [see D. D. Wall, Fibonacci series modulo m, *American Mathematical Monthly* 67 (1960) 525–532]. For example, it is easy to prove that each sequence is a single cycle, that is, the Fibonacci walk returns to the starting point $P_1 = (a, b)$. To see this, assume $P_s = P_t$ for some $s \neq t$ and s is the smallest positive integer for which such an equation holds. If $s > 1$, then the formula $f(n-1) = f(n+1) - f(n)$ for $n > 1$, applied to $n = s$ and to $n = t$, allows us to deduce that $P_{s-1} = P_{t-1}$, a contradiction; so it must be that $s = 1$.

The $m \times m$ grid is thus partitioned into orbits, each the cyclic Fibonacci walk generated by taking one of its points (a, b) as initial point P_1. A little experimentation shows that the cycle generated by $P_1 = (0, 1)$ plays a special role in the partition. In fact, this is the longest of the cycles and its length is divisible by the lengths of all other cycles! For if $\{u_n\}$ denotes the sequence $0, 1, 1, 2, \ldots (\mathrm{mod}\, m)$, with cycle length k, and $\{f(n)\}$ is the sequence $a, b, a+b, a+2b, \ldots (\mathrm{mod}\, m)$, with cycle length h, then

$$f(n) = au_{n-1} + bu_n \qquad \text{for } n > 1.$$

It follows that $f(n+k) = f(n)$, so $h | k$.

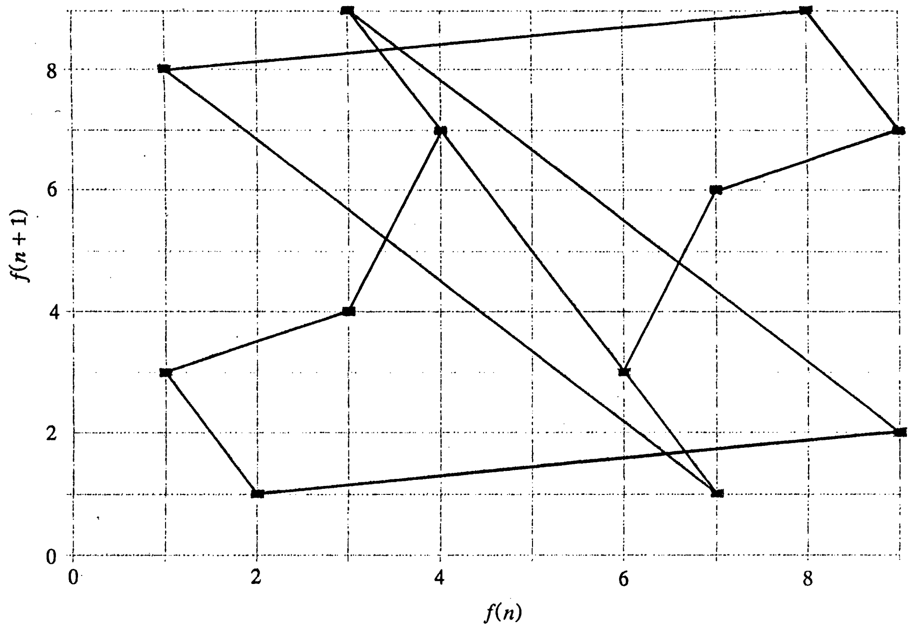

Figure 2. The Fibonacci walk for the sequence $1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, \ldots$.

The many questions to consider about the cycle lengths and orbit structures are a fertile source of projects requiring students to formulate and test conjectures.

Cubic Splines from Simpson's Rule

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Suppose at the points $x_0, x_1, \ldots, x_n$ we are given data values $y_0, y_1, \ldots, y_n$ and slopes $s_0, s_1, \ldots, s_n$. Then a *cubic Hermite interpolant* is a C^1 piecewise cubic curve $y = C(x)$ that interpolates these data values and slopes. In other words, on the data interval $[x_i, x_{i+1}]$ $C(x)$ is the unique cubic polynomial such that $C(x_i) = y_i$, $C'(x_i) = s_i$, $C(x_{i+1}) = y_{i+1}$, and $C'(x_{i+1}) = s_{i+1}$. It is easy to write down an explicit formula for $C(x)$ in each interval. Now suppose the slopes $s_0, s_1, \ldots, s_n$ are not given but are allowed to be chosen arbitrarily. It is a surprising fact that there is a choice of $s_0, s_1, \ldots, s_n$ that produces a cubic Hermite interpolant that is also C^2. Such an interpolant is called a *cubic spline*. It is shown in standard textbooks in numerical analysis that, for this to happen, $s_0, s_1, \ldots, s_n$ must satisfy the tridiagonal linear system

$$\begin{bmatrix} 1 & 0 & & & & \\ h_1 & 2(h_0+h_1) & h_0 & & & \\ & h_2 & 2(h_1+h_2) & h_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2(h_{n-2}+h_{n-1}) & h_{n-2} \\ & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_n \end{bmatrix}$$