which any one will be greater than one-half foot and the whole will be infinite."

Oresme's result was rediscovered by Pietro Mengoli in 1672 and Jacques Bernoulli in 1689 [**1**]. The proof given by Bernoulli in *Ars conjectandi* has been characterized by William Dunham [**3**] as "entirely different, yet equally ingenious." According to the *Oxford English Dictionary*, the name "harmonic series" first appeared in *Chambers Cyclopedia* (1727–51): "Harmonical series is a series of many numbers in continual harmonical proportion."

The roots of the harmonic series can be traced back to some of the earliest mathematical traditions in Western culture. Based on the harmonic mean, it is related to harmonics in both music and geometry. This concept deserves its place as a fundamental idea of mathematical thought.

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Factoring Quadratics

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Everyone knows that

$$x^2 + 5x + 6 = (x+2)(x+3) \quad \text{and}$$
$$x^2 + 5x - 6 = (x-1)(x+6).$$

Such pairs are not common, and not everyone is as familiar with

$$x^2 + 10x + 24 = (x+4)(x+6) \qquad x^2 + 13x + 30 = (x+3)(x+10)$$
$$x^2 + 10x - 24 = (x-2)(x+12) \qquad x^2 + 13x - 30 = (x-2)(x+15).$$

Here is how to find any number of such examples:

$$x^2 + Mx + N = (x + rt(s+t))(x + rs(s-t))$$
$$x^2 + Mx - N = (x - rt(s-t))(x + rs(s+t))$$

where r is any positive integer and s and t are relatively prime with opposite parity.

Readers may enjoy showing that these give all such pairs (details available from the author), and investigating the more general case, e.g.,

$$3x^2 + 29x + 70 = (3x + 14)(x + 5)$$
$$3x^2 + 29x - 70 = (3x + 35)(x - 2).$$

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Linear Transformation of the Unit Circle in R^2

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While studying linear transformations in R^2, it is customary to use the image of the unit square to illustrate the effect of the transformation and the relation between its determinant and the area of the image. We show that looking at the image of the unit circle yields an appealing and informative picture and also illustrates several basic ideas.

An invertible linear transformation always maps the unit circle U onto an ellipse. Suppose T is an invertible linear transformation with matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} \cos t \\ \sin t \end{pmatrix},$$

then

$$\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} dx - by \\ -cx + ay \end{pmatrix}.$$

The Pythagorean identity leads to the equation of the image of U,

$$\left(d^2 + c^2\right) x^2 - 2(ac + bd)xy + \left(a^2 + b^2\right) y^2 = (ad - bc)^2, \tag{1}$$

an ellipse centered at the origin. However, unless $ac + bc = 0$ its axes have been rotated away from the coordinate axes. To put (1) in standard form, we diagonalize the symmetric matrix of the quadratic form. Let

$$B = \begin{pmatrix} c^2 + d^2 & -(ac + bd) \\ -(ac + bd) & a^2 + b^2 \end{pmatrix} = (\det A)^2 \left(A^{-1}\right)^t \left(A^{-1}\right)$$

so $\det B = (\det A)^2 = (ad - bc)^2 = \lambda_1\lambda_2$, where λ_1 and λ_2 are the eigenvalues of B. If P is the matrix whose columns consist of the corresponding orthonormal eigenventors, let $\begin{pmatrix} x' \\ y' \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$. The equation of the ellipse may now be written as $\frac{x'^2}{\lambda_1} + \frac{y'^2}{\lambda_2} = 1$ and its area is $\pi\sqrt{\lambda_1\lambda_2} = \pi\ |ad - bc| = |\det A| \times$ (area of the unit disk).

For example, if $A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$, then $\det A = -6$, $T(x, y) = (2x + 2y, 2x - y)$, and the equation of TU is $5x^2 - 4xy + 8y^2 = 36$. So, $B = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$ has eigenvalues 4 and 9 with corresponding eigenvectors $(2/\sqrt{5} \quad 1/\sqrt{5})^t$ and $(-1/\sqrt{5} \quad 2/\sqrt{5})^t$.

