$p$ and $q$ from poll A and $\overline{x}_2$ and $\overline{y}_2$ are the estimates of $p$ and $q$ from poll B, both of the same size $n$, then $\overline{x}_1 - \overline{y}_1$ and $\overline{x}_2 - \overline{y}_2$ are each estimates of $p - q$. To describe the difference between these estimates we cannot simply add the individual margins of error. To do so would assume that the errors are in opposite directions, which is not always the case. Instead, consider that $E([\overline{X}_2 - \overline{Y}_2] - [\overline{X}_1 - \overline{Y}_1]) = 0$ and $\text{Var}([\overline{X}_2 - \overline{Y}_2] - [\overline{X}_1 - \overline{Y}_1]) = 2\text{Var}(\overline{X}_1 - \overline{Y}_1) \leq \frac{2}{n}$, since the results from the two polls are independent. The bound on the standard deviation is therefore $\sqrt{2}/\sqrt{n}$. This means that at least 95% of the time the two estimates of $p - q$ will differ by less than $2\sqrt{2}/\sqrt{n} = 2.8/\sqrt{n}$. If $p$ and $q$ are not too far from 1/2, then the estimates will differ by more than $2.8/\sqrt{n}$ about 5% of the time. Thus if polls A and B each have a margin of error of 3.5% (so $n \approx 900$), then about 5% of the time their estimates of the candidates' gap will differ by more than 10 percentage points. If many such polls are taken daily or weekly, then we can expect to see, every now and then, some pair of simultaneous polls that differ by 10 points or more in their estimates of the gap between the candidates.

———— o ————

## On Dividing Coconuts: A Linear Diophantine Problem

Sahib Singh (singh@vaxa.clarion.edu) and Dip Bhattacharya (bhattach@vaxa.clarion.edu), Clarion University of Pennsylvania, Clarion, PA 16214-2592

Paul Halmos gives the following problem in his recent book, *Problems for Mathematicians Young and Old* [MAA, Washington, DC, 1991, p. 22]:

> After gathering a pile of coconuts one day, five sailors on a desert island agree to divide them evenly after a night's rest. During the night one sailor gets up, divides the nuts into five equal piles with a remainder of one, which he tosses to a conveniently nearby monkey, and, secreting his pile, mixes up the others and retires. The second sailor does the same thing, and so do the third, the fourth, and the fifth. In the morning the remaining pile of coconuts (less one) is again divisible by 5. What is the smallest number of coconuts that the original pile could have contained?

Professor Halmos gives the solution in a very interesting way, using a fixed point theorem and negative eigenvalues. Here we give two other solutions that are more elementary. The first one involves linear equations, summation of geometric series, and logical reasoning. It is suitable for liberal arts students who are taking courses such as "Excursions in Mathematics." In the second method, we generalize the problem and use a little trick which avoids the summing of a geometric series.

**Method 1.** Let $N$ be the number of coconuts (assumed to be a whole number). Then according to the problem,

$$N = 5a + 1$$
$$4a = 5b + 1$$
$$4b = 5c + 1$$
$$4c = 5d + 1$$
$$4d = 5e + 1$$
$$4e = 5f + 1$$

where $a, b, \ldots$ are whole numbers. Eliminating $a, b, c, d$, and $e$ from the above equations:

$$
\begin{aligned}
N &= 1 + 5a \\
&= 1 + 5\left(\frac{1+5b}{4}\right) = 1 + \frac{5}{4} + \frac{5^2}{4}b \\
&= 1 + \frac{5}{4} + \frac{5^2}{4}\left(\frac{1+5c}{4}\right) = 1 + \frac{5}{4} + \left(\frac{5}{4}\right)^2 + \frac{5^3}{4^2}c \\
&\vdots \\
&= 1 + \frac{5}{4} + \left(\frac{5}{4}\right)^2 + \cdots + \left(\frac{5}{4}\right)^5 + \frac{5^6}{4^5}f = \frac{1-\left(\frac{5}{4}\right)^6}{-\frac{1}{4}} + \frac{5^6}{4^5}f \\
&= \frac{5^6}{4^5}(f+1) - 4.
\end{aligned}
$$

Since 4 and 5 are relatively prime, we conclude that $4^5$ must divide $(f+1)$. Thus the least value for $f+1$ is $4^5$ and, therefore, $N = 5^6 - 4 = 15621$.

**Method 2.** Generalize the problem and let $k$ be the number of sailors, where $k \geq 2$. We are to deal with the $k+1$ equations

$$
\begin{aligned}
N &= ka_1 + 1 \\
(k-1)a_1 &= ka_2 + 1 \\
(k-1)a_2 &= ka_3 + 1 \\
&\vdots \\
(k-1)a_k &= ka_{k+1} + 1,
\end{aligned}
$$

where $a_1, a_2, \ldots, a_{k+1}$ are natural numbers. Note that for each $m = 1, \ldots, k$,

$$
\begin{aligned}
a_m = \frac{ka_{m+1}+1}{k-1} &\Rightarrow a_m + 1 = \frac{ka_{m+1}+1}{k-1} + 1 \\
&\Rightarrow (a_m + 1) = \left(\frac{k}{k-1}\right)(a_{m+1}+1).
\end{aligned}
$$

This result is of recurring nature so we are able to write

$$(a_1 + 1) = \left(\frac{k}{k-1}\right)^k (a_{k+1}+1).$$

Thus,

$$N = ka_1 + 1 = k(a_1+1) - k + 1 = \frac{k^{k+1}}{(k-1)^k}(a_{k+1}+1) - k + 1.$$

Being consecutive integers, $k$ and $k-1$ are relatively prime, so the minimum value of $N$ is achieved when $a_{k+1} + 1 = (k-1)^k$. Therefore,

$$N = k^{k+1} - (k-1).$$

In particular, when $k = 5$, $N = 5^6 - 4$ as in method 1.