inequalities, showing that (roughly speaking) the area enclosed by a circle of radius r in Thurston paper grows exponentially as r increases. The particular exploration you might assign would depend on the goals of your course, but in any event, after working with Thurston paper your students should have sufficient insight and motivation to begin a more rigorous discussion of hyperbolic geometry.

References

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t-Probabilities as Finite Sums

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A text **[1]** has an exercise to derive the *t*-probability formula,

$$P[t \geq R, df = N] = \frac{\Gamma((N+1)/2)}{\sqrt{N\pi}\,\Gamma(N/2)} \int_R^\infty (1 + x^2/N)^{-(N+1)/2}\, dx \equiv P_N,$$

where *df* denotes the number of degrees of freedom. The purpose of this note is to show how this expression may be written as a finite sum and thus may be evaluated by writing a program on any programmable calculator.

Substitute $x = N^{1/2} \tan\theta$ and define φ by $R = N^{1/2} \tan\varphi$. Then P_N becomes

$$P_N = \frac{\Gamma((N+1)/2)}{\sqrt{\pi}\,\Gamma(N/2)} \int_\varphi^{\pi/2} \cos^{N-1}\theta\, d\theta.$$

Since φ depends on N, define for $K \geq 1$

$$Q_K = Q_K(N) \equiv \frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)} \int_\varphi^{\pi/2} \cos^{K-1}\theta\, d\theta$$

and note that $P_N = Q_N(N)$. In order to compute P_N we use the reduction formula

$$\int \cos^m\theta\, d\theta = \frac{1}{m}(\sin\theta)(\cos^{m-1}\theta) + \frac{m-1}{m}\int \cos^{m-2}\theta\, d\theta.$$

Since Q_1 and Q_2 can be computed directly, we have for $K \geq 3$

$$Q_K = -\frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)} \frac{1}{K-1}(\sin\varphi)(\cos^{K-2}\varphi) + \frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)} \frac{K-2}{K-1} \int_\varphi^{\pi/2} \cos^{K-3}\theta\, d\theta \tag{1}$$

The properties of the gamma function,

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1, \quad \text{and} \quad \Gamma(1/2) = \sqrt{\pi},$$

may be used to show that

$$\frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)} \frac{K-2}{K-1} \int_{\varphi}^{\pi/2} \cos^{K-3}\theta \, d\theta = Q_{K-2}.$$

Moreover,

$$\begin{aligned} J_K = J_K(N) &\equiv \frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)} \frac{1}{K-1} (\sin\varphi)(\cos^{K-2}\varphi) \\ &= \frac{\Gamma((K-1)/2)}{2\sqrt{\pi}\,\Gamma(K/2)} \frac{R}{\sqrt{R^2+N}} \left(\frac{N}{R^2+N}\right)^{(K-2)/2} \end{aligned}$$

and it follows that

$$J_{K+2}(N) = \frac{K-1}{K} \frac{N}{R^2+N} J_K(N) \quad \text{for } K \geq 3.$$

From (1), we now have

$$Q_K = -J_K + Q_{K-2}, \quad K \geq 3. \tag{2}$$

Since this relates Q_K to Q_{K-2}, we examine Q_N for N even and N odd. If N is even, say $N = 2L$, then repeated use of (2) yields

$$P_N = Q_N = Q_2 - \sum_{I=2}^{L} J_{2I}.$$

Since

$$Q_2 = \frac{1}{2}\left(1 - \frac{R}{\sqrt{R^2+N}}\right), \quad J_4 = \frac{1}{4} \frac{N}{R^2+N} \frac{R}{\sqrt{R^2+N}}, \quad \text{and} \quad J_{2I+2} = \frac{2I-1}{2I} \frac{N}{R^2+N} J_{2I},$$

a simple program can be written to evaluate P_N if N is even.

If N is odd, say $N = 2L - 1$, then

$$P_N = Q_N = Q_I - \sum_{I=2}^{L} J_{2I-1}.$$

Since

$$Q_1 = \frac{1}{2} - \frac{1}{\pi} \tan^{-1}\left(\frac{R}{\sqrt{N}}\right), \quad J_3 = \frac{1}{\pi} \frac{R\sqrt{N}}{R^2+N^2} \quad \text{and} \quad J_{2I+1} = 2\frac{I-1}{2I-1} \frac{N}{R^2+N} J_{2I-1},$$

we can now program the case where N is odd.

The program I wrote for a TI-85 calculator computed, to six places,

$$P[t > 2.132, df = 4] = 0.049991 \quad \text{and} \quad P[t > 4.032, df = 5] = 0.005001.$$

Standard t-tables show these values as 0.05 and 0.005, respectively.

Reference

1. Dennis Wackerly, William Mendenhall, and Richard Scheaffer, *Mathematical Statistics with Applications*, 5th ed., Wadsworth, 1996.

