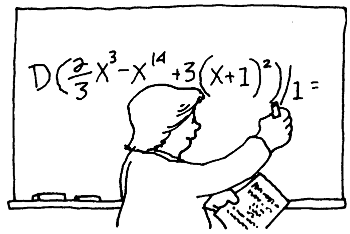
CLASSROOM CAPSULES



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A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

Theory vs. Computation in Some Very Simple Dynamical Systems

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The purpose of this note is to illustrate, via two numerical experiments, the great disparity that may exist between theoretical and computational results in even the most primitive of iterative systems. Consider these functions from $[0,1]$ to $[0,1]$:

1) $g(x) = 2x(\text{mod } 1)$, i.e. $g(x)$ is the fractional part of $2x$.
2) $h(x) = x + 1/2$ if $0 \le x \le 1/2$, $h(x) = 2(1-x)$ if $1/2 \le x \le 1$.
(See Figure 1.)

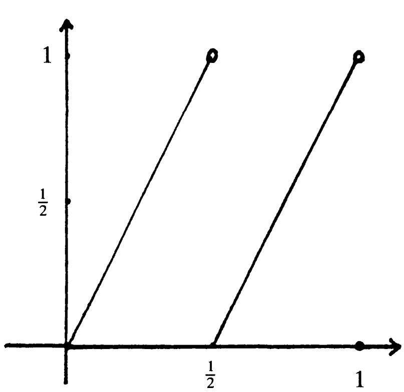

Figure 1a
Graph of g.

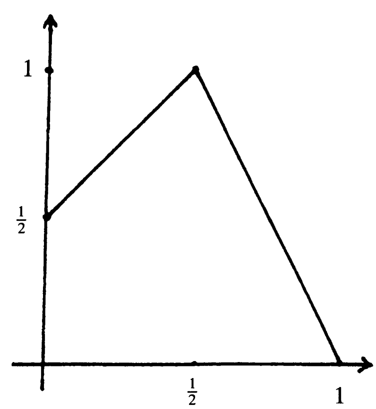

Figure 1b
Graph of h.

Experiment 1 is to write and execute a computer program that generates, for arbitrary $x \in [0,1]$, 100 terms of the sequence $x_0 = x$, $x_n = g(x_{n-1})$ for $n \ge 1$. Experiment 2 is to do the same thing for h. For example, if $x = 2/3$, the sequence for g should be $2/3, 1/3, 2/3, 1/3, \ldots$ and the sequence for h should be $2/3$,

$2/3, 2/3, \ldots$. However, the computer output may be surprising—try these experiments before reading further!

Theory—Where are the periodic orbits? "Dynamical systems" may be defined as the study of iterations of functions $f\colon S \to S$, where S is some set. For any $x \in S$, the orbit of x is defined to be the sequence $x_0 = x$, $x_n = f(x_{n-1})$ for $n \geq 1$. Orbits may behave with various degrees of regularity or irregularity; the central problem of dynamical systems is to classify this behavior. Periodic orbits are of particular interest. The orbit of x is said to be n-periodic if $x_0 = x_n$ but $x_0 \neq x_k$ for $k = 1, 2, \ldots, n-1$. If *some* term of the orbit of x has an n-periodic orbit, then the orbit of x is said to be eventually n-periodic. If the value of n is unspecified, we refer simply to "periodic" or "eventually periodic" orbits.

Let us examine the orbits relative to g. It is easy to see that each orbit consists entirely of rationals or entirely of irrationals. Furthermore, the eventually periodic orbits are precisely the rational ones. To prove this, first note that if x is a rational of form p/r, then all terms of the orbit of x have form q/r (not necessarily in lowest terms). Since there are only $r+1$ such rationals in $[0,1]$, there must be some repetition, and the result follows directly. Conversely, if x has an eventually n-periodic orbit then some term must be a root of $g^n(y) - y = 0$, where g^n denotes the n-fold composition of g with itself. But $g^n(y) - y$ is piecewise linear with integer coefficients and so has rational roots.

Now, if x is a rational of the form $p/2^k q$, $k \geq 1$, with q odd then $g(x)$ has the form $r/2^{k-1}q$, i.e., g cancels twos from denominators. This suggests that it may be illuminating to look at the orbits relative to g in binary notation. Recall that every number in $[0,1]$ can be represented in the form $.\beta_1\beta_2\beta_3\ldots (= \beta_1/2 + \beta_2/4 + \beta_3/8 + \cdots)$, where the β's are in $\{0,1\}$. This representation is unique except for the usual stipulation about infinite strings of 1's. We have then $g(.\beta_1\beta_2\beta_3\ldots) = .\beta_2\beta_3\ldots$, and this fact renders the dynamics of the orbits transparent. In particular, the orbit of x will be eventually periodic precisely when $x = .\beta_1\ldots\beta_k\overline{\beta_{k+1}\ldots\beta_{k+n}}$. (The overline indicates infinite repetition of the block.) These are, again, just the rationals. The points with n-periodic orbits are those with binary representations of form $.\overline{\beta_1\beta_2\ldots\beta_n}$, except when the overlined block itself consists of a repeated sub-block. For example, the orbit of $x = .\overline{100\ldots0}$ (length n) is n-periodic. The reader may verify that $x = 2^{n-1}/(2^n - 1)$.

We now turn to our second example, h. This is slightly more subtle and is included lest the reader think that the numerical pathology of g is due entirely to its discontinuity. Details are left as exercises.

Exercise 1. Prove that x has an eventually periodic orbit relative to the function h if and only if x is rational. Hint: if x has form p/q, then all terms in the orbit of x can be written in the form $r/2q$.

Exercise 2. Given an arbitrary positive integer n, find a point x with an n-periodic orbit relative to h. Hint: the action of h in binary is $h(.\beta_1\beta_2\beta_3\ldots) = .1\beta_2\beta_3\ldots$ if $\beta_1 = 0$, and $h(.\beta_1\beta_2\beta_3\ldots) = .\beta_2'\beta_3'\ldots$ if $\beta_1 = 1$. Here $\beta_k' = 1$ if $\beta_k = 0$ and $\beta_k' = 0$ if $\beta_k = 1$. Using this, construct an x for which $x_0, \ldots, x_{n-2}$ have leading digit 1 and x_{n-1} has leading digit 0. The cases n even and n odd should be examined separately. (The expert will recognize that the existence of n-periodic orbits for all n is guaranteed by Sarkovskii's theorem. However, this theorem does not tell us how to get them explicitly. See [Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Addison-Wesley, Menlo Park, CA, 1989] for a discussion.)

Computation—Why does the computer lie? The reader will have observed that no matter what x is, the computer output soon becomes $\cdots 0, 0, 0, \cdots$ for g and $\cdots 0$, $1/2$, 1, 0, $1/2$, $1 \cdots$ for h. What has happened to all the other orbits? The answer lies in the fact that virtually all computers do arithmetic in binary, and so the only numbers in $[0, 1]$ that can be represented exactly are those of the form $p/2^k$, $0 \le p \le 2^k$, with some upper bound on k depending on the type of computer. Since $g\ (p/2^k)$ always has the form $r/2^{k-1}$, $0 \le r < 2^{k-1}$, after at most k iterations the computed result will be 0 repeated forever. Now consider the action of h on $p/2^k$. It is easily verified that if $1/2 \le p/2^k \le 1$, then $g(p/2^k)$ is expressible in the form $r/2^{k-1}$, $0 \le r \le 2^{k-1}$. Since no two successive terms of any orbit of h lie in $[0, 1/2)$, it follows that at least every second iteration of h results in a cancellation of a 2 from the denominator. Once again, the computer output will soon be 0, followed by infinite repetition of $1/2$, 1, 0.

A glimpse of chaos. There is no universally agreed-upon definition of a chaotic system. Following Devaney, we say that a function f: $[0, 1] \to [0, 1]$ has chaotic dynamics if these three conditions are satisfied:

i) For every $x \in [0, 1]$, there is a y, arbitrarily close to x, for which the orbits of x and y eventually diverge by some preassigned distance. Precisely, there is a $\delta > 0$ such that for every $x \in [0, 1]$ and every $\epsilon > 0$, there exist $y \in [0, 1]$ and a positive integer m for which $|x - y| < \epsilon$ and $|x_m - y_m| > \delta$. This property is called sensitive dependence on initial conditions.
ii) The set of points lying on periodic orbits is dense. (A set $D \subset [0, 1]$ is dense if for every $x \in [0, 1]$ and every $\epsilon > 0$, there is a point $d \in D$ such that $|x - d| < \epsilon$).
iii) There is a dense orbit.

Exercise 3. Both g and h have chaotic dynamics. Hint: use binary notation. The points $x = .\beta_1\beta_2\beta_3 \cdots$ and $y = .b_1b_2b_3 \cdots$ are close if $\beta_i = b_i$ for $i = 1, 2, \ldots, n$; the larger n is, the closer they are. The proof for g is straightforward. The proof for h involves some more delicate manipulation.

What to do with a calculator. Some hand calculators use a binary-coded decimal system and perform decimal arithmetic exactly. The user of such a calculator may enjoy examining the functions

3) $\bar{g}(x) = 10x \pmod 1$.
4) $\bar{h}(x) = x + 1/10$ if $0 \le x \le 9/10$, $\bar{h}(x) = 10(1 - x)$ if $9/10 \le x \le 1$.