A Geometric Interpretation of Simpson's Paradox

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In his interesting capsule, Knapp [CMJ 16 (June 1985) 209–211] illustrates Simpson's Paradox with the following fictitious example:

	Player A			Player B		
	Times at bat	Hits	Average	Times at bat	Hits	Average
Against right-handed pitchers	202	45	.223 $=(A_r)$	250	58	.232 $(=B_r)$
Against left-handed pitchers	250	71	.284 $(=A_l)$	108	32	.296 $(=B_l)$
Overall	452	116	.257 $(=A)$	358	90	.251 $(=B)$

In this example, Player A has a better overall batting average than Player B despite the fact that Player A's average was worse than Player B's against both right-handed pitchers and left-handed pitchers. The key to the paradox is the differential weighings

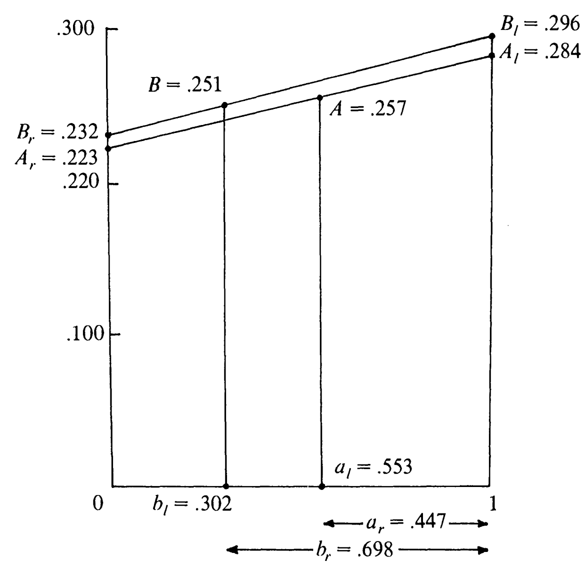
$$a_r = \frac{202}{452} = .447 \quad a_l = \frac{250}{452} = .553 \qquad \text{and} \qquad b_r = \frac{250}{358} = .698 \quad b_l = \frac{108}{358} \doteq .302$$

of the right- and left-averages when the overall averages are computed. Since

$$A = a_r A_r + a_l A_l \quad \text{and} \quad B = b_r B_r + b_l B_l,$$

there is no necessary ordering of A and B. As shown here, the selection of these relative weights sufficed to switch the rank-order of the overall hitting averages of the two players.

As Knapp indicated, Simpson's Paradox is a nonintuitive phenomenon that does not occur very often and is seldom taught in the classroom. Here we show that the "paradox" becomes more transparent if one uses the geometrical construction of Hoehn [CMJ 15 (March 1984) 135–139] for the weighted mean. The figure below depicts the above example.



Although $A_r < B_r$ and $A_l < B_l$, we "see" that $A > B$ (draw horizontal lines through A and B)!

Editor's Note: Suppose the line $y = B_r$ intersects the line $L_A(x) = (A_l - A_r)x + A_r$ at the point (a, B_r). Then for any fixed value $a_l \in (a, 1)$, the line $y = L_A(a_l)$ intersects the line $L_B(x) = (B_l - B_r)x + B_r$ at a point whose abscissa is $b \in (0, a_l)$, and Simpson's Paradox occurs for all $b_l \geq b$.

Does "holds water" Hold Water?

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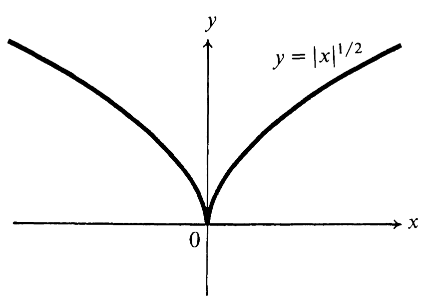
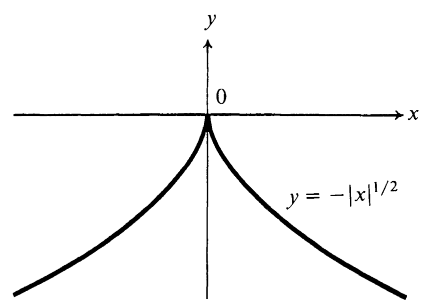
A number of calculus books give the mnemonic that a curve which is concave upward "holds water," whereas one which is concave downward "spills water." Recently, a student asked one of my colleagues why the graph of $|x|^{1/2}$ is not concave upward in an interval containing 0, because it would evidently hold water. Indeed, there are actual glasses that have a similar shape (with the addition of a stem) and do hold wine (if not water). The problem arises because the mnemonic, like most mnemonics, is flawed: "holding water" is neither necessary nor sufficient for a given curve to be concave upward. The student's example can be augmented by $y = -|x|^{1/2}$, which clearly "spills water."





The late Professor W. R. Ransom used to use "bowl shaped" and "dome shaped." I have often followed his lead without realizing that this mnemonic is equally flawed. Wide bowls with sides that curve downward are not uncommon, and the domes on Eastern Orthodox churches are not concave downward at the extreme top. I have also seen the mnemonic that a smile is concave up and a frown (or scowl) is concave down; this connects nicely with the colloquial use of "up" and "down." Here smiles are to be thought of as in cartoons and graffiti; actual smiles can be misleading, especially when they are crooked.

I am indebted to A. M. Trimbinska for telling me about the puzzled student.