games. Equating these expressions for the total number of games yields

$$1+t+t^2+\cdots+t^{n-1}=\frac{t^n-1}{t-1}. \tag{1}$$

Although we derived (1) by considering only integral $t>1$ and positive integral n, it forms an algebraic identity valid for all complex $t \neq 1$. Replacing n with $n+1$ gives the more standard form

$$1+t+t^2+\cdots+t^n=\frac{t^{n+1}-1}{t-1},$$

valid for all whole numbers n.

Weighted Means of Order r and Related Inequalities: An Elementary Approach

François Dubeau, Collège militaire royal de Saint-Jean, Saint-Jean-sur-Richelieu, Québec, Canada, J0J 1R0

The aim of this note is to present the properties of weighted means of order r using elementary techniques of analysis. In particular the increasing property of the weighted mean of order r, as a function of r, is proved using a more elementary technique than the standard proof.

Let r be any nonzero real number and let us consider the function $\phi(x)=x^r$ defined on the interval $I=(0,+\infty)$. Using the mean value theorem, for any fixed strictly positive real number a, we have

$$x^r=a^r+r\xi_a^{r-1}(x-a) \tag{1}$$

where ξ_a is between x and a. It follows that $\phi(x)=x^r$ is strictly increasing (decreasing) if $r>0$ $(r<0)$.

Using the Taylor expansion of order 1, we have

$$x^r=a^r+ra^{r-1}(x-a)+r(r-1)\xi_a^{r-2}\frac{(x-a)^2}{2} \tag{2}$$

where ξ_a is between x and a. Set $x=x_i>0$ in (2), multiply by $\alpha_i>0$ and sum for $i=1,\ldots,n$. Without loss of generality let us assume that $\Sigma_{i=1}^n\alpha_i=1$ and let us use the notation $\Sigma\beta_i$ for $\Sigma_{i=1}^n\beta_i$. Set $a=\Sigma\alpha_ix_i>0$ in (2). It follows that

$$\sum\alpha_ix_i^r\left\{\begin{matrix}\geq\\ \leq\end{matrix}\right\}\left(\sum\alpha_ix_i\right)^r \quad \text{for} \begin{cases} r<0 \quad \text{or} \quad r>1, \\ 0<r<1. \end{cases} \tag{3}$$

Equality holds in (3) iff the x_i's are all equal. Relation (3) shows that $\phi(x)=x^r$ is strictly convex (concave) for $r<0$ or $r>1$ $(0<r<1)$.

Using the increasing or decreasing property of $\phi(x)=x^r$ we obtain from (3)

$$\sum\alpha_ix_i\left\{\begin{matrix}\leq\\ \geq\end{matrix}\right\}\left(\sum\alpha_ix_i^r\right)^{1/r} \quad \text{for} \begin{cases} r>1, \\ r<1, \quad r\neq 0, \end{cases} \tag{4}$$

with equality iff the x_i's are all equal.

The relation (4) contains several well known inequalities.

Example 1. Set $\alpha_i = 1/n$. If $r = 2$ we have

$$\frac{1}{n}\sum x_i \le \left(\frac{1}{n}\sum x_i^2\right)^{1/2}$$

which is the arithmetic mean-root mean square inequality. If $r = -1$ then $(1/n)\Sigma x_i \ge ((1/n)\Sigma 1/x_i)^{-1}$ which is the harmonic mean-arithmetic mean inequality.

Example 2. Let $r \ne 0$ and s be such that $1/r + 1/s = 1$. Let $\eta_i > 0$ and $\xi_i > 0$, and set $\alpha_i = \eta_i^s/(\Sigma\eta_i^s)$ and $x_i = \xi_i/\eta_i^{s/r}$. It follows that

$$\sum \xi_i\eta_i \begin{Bmatrix} \le \\ \ge \end{Bmatrix} \left(\sum \xi_i^r\right)^{1/r}\left(\sum \eta_i^s\right)^{1/s} \quad \text{for} \begin{cases} r > 1, \\ r < 1, \quad r \ne 0, \end{cases}$$

which is the classical inequality of Hölder. For $r = s = 2$ we obtain the Cauchy-Schwarz inequality.

Let $r \ne 0$ and consider $M_r(x, \alpha) = (\Sigma\alpha_i x_i^r)^{1/r}$ as a function of r. Let us prove two properties of this function.

Property 1. *$M_r(x, \alpha)$ is a strictly increasing function with respect to r iff the x_i's are not all equal.*

Proof. Consider $\xi_i > 0$ $(i = 1, \dots, n)$ and assume $s < r$. If $r > 0$ then $s/r < 1$ and from (4) we have

$$\sum \alpha_i\xi_i \ge \left(\sum \alpha_i\xi_i^{s/r}\right)^{r/s}.$$

If $r < 0$ then $1 < s/r$ and again from (4) we have

$$\sum \alpha_i\xi_i \le \left(\sum \alpha_i\xi_i^{s/r}\right)^{r/s}.$$

Set $\xi_i = x_i^r$ and consider the rth root. We obtain

$$\left(\sum \alpha_i x_i^s\right)^{1/s} < \left(\sum \alpha_i x_i^r\right)^{1/r}. \quad \square$$

Remark 1. The standard proof of this result presented by Cooper [**2**], and that we can also find in [**1**, pp. 16–18] or [**3**, pp. 76–77], is based on the convexity of $x \log x$ or the log-convexity of x^x.

Let us use the notation $\underline{\xi} = \min\{\xi_1, \dots, \xi_n\} \le \max\{\xi_1, \dots, \xi_n\} = \bar{\xi}$.

Property 2. *The following limits hold for $M_r(x, \alpha)$:*

(i) $$\lim_{r \to +\infty} M_r(x, \alpha) = \bar{x}$$

(ii) $$\lim_{r \to 0} M_r(x, \alpha) = \Pi x_i^{\alpha_i},$$

(iii) $$\lim_{r \to -\infty} M_r(x, \alpha) = \underline{x}.$$

Proof of (i). For any $r > 0$ we have $\underline{\alpha}^{1/r}\bar{x} \le M_r(x, \alpha) \le \bar{x}$ and the result follows from $\lim_{r \to +\infty} \underline{\alpha}^{1/r} = 1$.

Proof of (ii). We have $M_r(x, \alpha) = \exp((1/r)\log(\Sigma \alpha_i x_i^r))$ and using the mean value theorem we have

$$\log\left(\sum \alpha_i x_i^r\right) = r\frac{\sum \alpha_i x_i^\rho \log x_i}{\sum \alpha_i x_i^\rho}$$

where ρ is strictly between 0 and r. When $r \to 0$ then $\rho \to 0$ and the result follows (we can also prove this result using l'Hospital's rule [**1**, p. 16]).

Proof of (iii). For $r < 0$ we have $M_r(x, \alpha) = 1/M_{-r}(1/x, \alpha)$ and the result follows from (i). □

In summary we have for $s < 0 < r$

$$\underline{x} \le \left(\sum \alpha_i x_i^s\right)^{1/s} \le \Pi x_i^{\alpha_i} \le \left(\sum \alpha_i x_i^r\right)^{1/r} \le \bar{x} \tag{5}$$

with equalities iff the $x_i's$ are all equal.

Example 3. For $r = 1$ and $\alpha_i = 1/n$ we obtain $(\Pi x_i)^{1/n} \le (1/n)\Sigma x_i$ which is the arithmetic mean-geometric mean inequality.

Finally, inequalities (5) can be extended to integrable functions. For example, consider a Riemann integrable function $f(x)$ defined on $[a, b]$ and such that $0 < m \le f(x) \le M < +\infty$. If we set $\alpha_i = 1/n$ and $x_i = f(\xi_i)$, where $a + (i-1)((b-a)/n) \le \xi_i \le a + i((b-a)/n)$, and consider the limit when n goes to $+\infty$ we obtain for $s < 0 < r$

$$m \le \left(\frac{1}{b-a}\int_a^b f^s(x)\,dx\right)^{1/s} \le e^{(1/b-a)\int_a^b \log f(x)\,dx} \le \left(\frac{1}{b-a}\int_a^b f^r(x)\,dx\right)^{1/r} \le M.$$

We can obtain the same result for a Lebesque integrable function.

References

1. Beckenbach, E. F. and Bellman R., *Inequalities*, Springer-Verlag, Berlin-Heidelberg-New York, 1965.
2. Cooper, R., Notes on certain inequalities: II, *Journal of the London Mathematical Society* 2 (1927) 159–163.
3. Mitrinovic, D. S., *Analytic Inequalities*, Springer-Verlag, New York-Heidelberg-Berlin, 1970.

In My Experience...

Experience is the name everyone gives to their mistakes.

Oscar Wilde, *The Oxford Dictionary of Quotations*, Third Ed., Oxford University Press, N.Y., 1979, p. 573.