unknown weight up to 121 pounds with a combination of stones placed on both sides of the scale.

Finally, we mention that the woman with five stones is able to accomplish an even more impressive feat.

> The woman claims that, with the aid of a two-pan balance, she can determine the weight of any rock you give her provided it is of integral weight of at most 242 pounds! What are the weights of her five stones?

The trick here is to double all the weights of the previous puzzle: the woman uses stones of 2, 6, 18, 54, and 162 pounds. As before, she can balance the weight of any rock you hand her with a combination of stones placed on both sides of the balance, provided your rock weighs an even number of pounds. If it doesn't, she can, at the very least, determine that your rock weighs more than one even number and less than the next consecutive even number. From this, she then knows the weight of your rock!

At this point, students may be ready and motivated to consider more challenging related problems, for example:

> How many seemingly identical coins can be tested on a two-pan balance in order to determine whether the coins are equally weighted—and, if not, to identify which is the unique heavy or light coin?

The answer, of course, depends on the number of weighings allowed. Three coins can be tested in two weighings, and up to twelve coins can be tested in three weighings. In general, k ($k \geq 2$) weighings suffice to test any set up to $(3^k - 3)/2$ coins. For a proof with examples, and related references, see [**1**].

Reference

1. Aaron L. Buckman and Frank S. Hawthorne, Coin Weighings Revisited, in *Two-Year College Mathematics Readings* (Warren Page, editor), Mathematical Association of America (1981) 275–281.

Convergence-Divergence of *p*-Series

Xianfu Wang (xwang@uccb.ns.ca), University College of Cape Breton, Sydney, Nova Scotia, Canada

Recently, Khan [**1**] proved the divergence and convergence of the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \tag{1}$$

by rearrangement of the series rather than by the integral test. This capsule illustrates that the p-series divergence-convergence essentially follows from knowlege about the geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots, \tag{2}$$

which converges if $|r| < 1$, and diverges if $|r| \geq 1$.

When $p = 1$, the p-series diverges. Observe that

$$\frac{1}{3} + \frac{1}{4} \geq 2\frac{1}{4} = \frac{1}{2},$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 4\frac{1}{8} = \frac{1}{2},$$
$$\vdots$$
$$\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n} \geq 2^{n-1}\frac{1}{2^n} = \frac{1}{2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} + \cdots.$$

diverges. This also follows from (2).

When $p > 1$, the p-series converges. Observe that

$$\frac{1}{2^p} + \frac{1}{3^p} \leq 2\frac{1}{2^p} = \frac{1}{2^{p-1}},$$
$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \leq 4\frac{1}{4^p} = \left(\frac{1}{2^{p-1}}\right)^2,$$
$$\vdots$$
$$\frac{1}{(2^n)^p} + \cdots + \frac{1}{(2^{n+1}-1)^p} \leq 2^n\frac{1}{(2^n)^p} = \left(\frac{1}{2^{p-1}}\right)^n,$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq 1 + \frac{1}{2^{p-1}} + \cdots + \left(\frac{1}{2^{p-1}}\right)^n + \cdots. \tag{3}$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges because $1/2^{p-1} < 1$ and (2) converges.

[Note: The function $f : (1, +\infty) \to R$ given by

$$f(p) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p,$$

is convex and decreasing because each term is convex and decreasing with respect to p. Since

$$1 < f(p) \leq \frac{1}{1 - 1/2^{p-1}},$$

we have $\lim_{p\to\infty} f(p) = 1$.]

The divergence and convergence of $\sum_{n=1}^{\infty} 1/n^p$ also follows from the Cauchy Condensation Test [**2**, p. 714, ex. 32]: Let $\{a_n\}$ be a nonincreasing sequence of positive terms that converge to zero. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

References

1. R. A. Khan, Convergence-divergence of p-series, *The College Mathematics Journal* **32** (2001) 206–208.
2. R. L. Finney, G. B. Thomas, F. D. Demana, and B. K. Waits, *Calculus*, Addison-Wesley, 1994.

Observations on the Indeterminacy of the Sample Correlation Coefficient

Owen Byer (byer@emu.edu), Eastern Mennonite University, Harrisonburg, VA 22802-2462

If you've taught an introductory statistics course, you've probably introduced the linear correlation coefficient

$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\sum x^2 - \left(\sum x\right)^2}\sqrt{n\sum y^2 - \left(\sum y\right)^2}} \tag{1}$$

by stating something along the lines of "it measures the strength of the linear relationship between paired x and y values." Students generally easily understand that $-1 \le r \le 1$, with increasingly large values of $|r|$ corresponding to increasingly stronger correlation. However, the question "What is the value of r if all of the y-values are equal?" prompted some interesting discussions and surprising results in a recent course I taught. In this capsule, we will recount some simple, yet nonintuitive, phenomena about correlation that arose while we explored some answers to this question.

It is known that the sample correlation coefficient r cannot be computed if all the y-values or all the x-values are equal [D. Sheskin, *Handbook of Parametric and Nonparametric Statistical Procedures*, CRC Press (1997) 549–550]. Indeed, if $y_i = c$ for all i, then the formula above for r results in the indeterminate form $\frac{0}{0}$. Attempts to use intuition to define r in this case are equally unsuccessful. If one focuses on the word *linear* in the first sentence above, then the answer to the question could be 1. Of course, there is no reason that the correlation should be positive, so one could argue for $r = -1$ just as well. On the other hand, if one focuses on the phrase *strength of the relationship between paired x and y*, then it is easy to argue that $r = 0$ since for any (x, y) pair, the x-value seems to have no bearing on the value of y.

As any good calculus student knows, however, some discontinuities are removable. Perhaps this happens here. Certainly if one of the y_i values is altered ever so slightly from the others, the formula for r will be valid and we won't have changed the correlation much, right? We tried this as a class using the software *Excel*. For the six pairs of data $\{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2.01)\}$ the linear correlation was $0.65465367\ldots$.

We intended to look for a limiting value for r as y_6 approached 2, while $y_i = 2$ for $i = 1, 2, \ldots, 5$. Much to our surprise, we found that the correlation remained