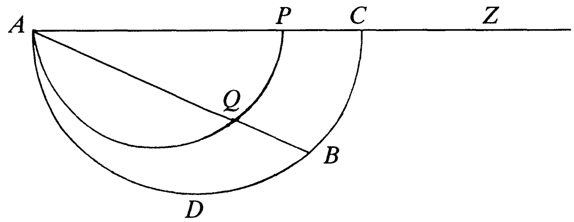

Figure 4

> AQP meeting the straight line AB (assumed drawn and produced if necessary) in the point Q, and further a second cycloid ADC whose base and height are to the base and height of the former as AB to AQ respectively. This last cycloid will pass through the point B, and it will be that curve along which a weight, by the force of its gravity, shall descend most swiftly from the point A to the point B.

The analytic version of this construction is to let $C(k)$ be any cycloid in standard position and let Q be the point where the first arch of $C(k)$ intersects the line AB. The coordinates of Q have the form (rx_1, ry_1) for some positive number r. Now the first arch of $C(k/r)$ meets B because

$$rx_1 = k(\theta - \sin\theta) \qquad \text{implies} \qquad x_1 = (k/r)(\theta - \sin\theta)$$

and similarly for y_1.

For a leisurely exposition of Johann Bernoulli's ingenious solution of the brachistochrone problem, see [**3**]. A thorough history of the problem may be found in [**2**].

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Another Way to Graph a Sequence

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One way to graph a sequence $(a_1, a_2, a_3, \ldots)$ is to plot the points $(1, a_1), (2, a_2), (3, a_3), \ldots$. This common approach emphasizes the fact that the sequence is a function of the counting numbers. Graphically, the beginning of the sequence is emphasized and the tail of the sequence is off the right-hand side of the page (or blackboard). However, the tail is often the part of greatest interest, so an alternative graph that emphasizes the tail is useful.

A plot of the points $(1/1, a_1), (1/2, a_2), (1/3, a_3), \ldots, (1/n, a_n), \ldots$ provides a clear picture of the tail. Note that the tail of the sequence is now on the left, near the y-axis. The limit of the sequence (a_n), if it exists, is the y-coordinate of the unique limit point of this set of points. In Figure 1, a graph of the sequence $a_n = 1 + (-1)^n/n^2$, the ink emphasizes the tail of the sequence, and the limit is the y-value of the limit point on the y-axis. By drawing the lines $y = 1 \pm \epsilon$, we can easily illustrate that the *tail* of the sequence lies between the upper and lower ϵ-tolerances. To illustrate a tighter tolerance, we simply "zoom in" on the part of the graph near $(0, 1)$ to indicate what happens later in the sequence; see Figure 2.

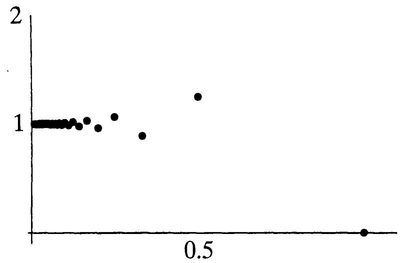

Figure 1

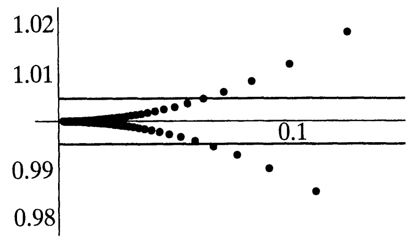

Figure 2

This representation also deftly illustrates the limit superior and limit inferior of a sequence. Figure 3, the graph of the sequence $a_n = 2/5 + (n + 50)/n \sin(3n)$, suggests that the limit superior is $7/5$. With this guidance it is straightforward to write out a proof of that fact. This type of graph can be used to indicate how to

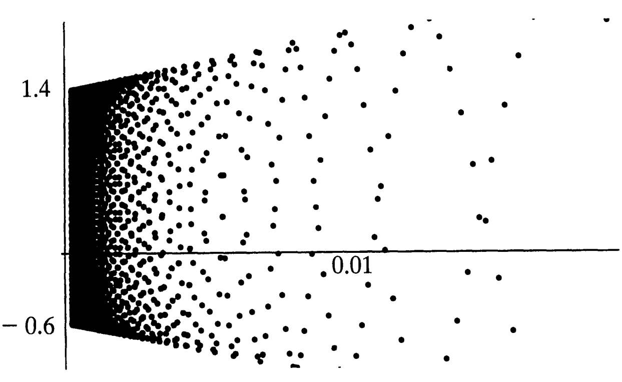

Figure 3

choose a subsequence that converges to the limit superior: Identify a reasonable sample of points converging to the point $(0, 7/5)$ and then invert the x-coordinates to identify the elements of the subsequence.