Characterizing Power Functions by Volumes of Revolution

Bettina Richmond (bettina.richmond@wku.edu) and Tom Richmond (tom.richmond @wku.edu), Western Kentucky University, Bowling Green, KY 42101-3576

Given a nonnegative increasing function f defined for $x > 0$, let $V(r)$ denote the volume of the solid obtained by revolving about the y-axis the first quadrant region under the curve $y = f(x)$ for $0 \leq x \leq r$, and let $C(r) = \pi r^2 f(r)$ be the volume of the corresponding right circular cylinder with radius r and height $f(r)$, as shown in Figure 1.

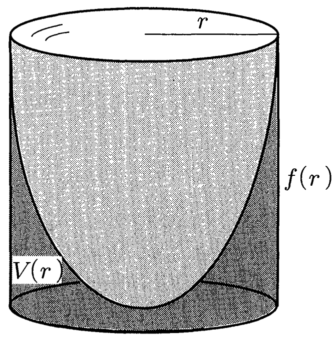

Figure 1. Inner cavity has volume $C(r) - V(r)$.

When $f(x) = kx$ the inner cavity is a cone; thus $V(r)/C(r) = \frac{2}{3}$. If $f(x) = kx^2$ the cavity is a paraboloid with volume equal to $V(r)$, so $V(r)/C(r) = \frac{1}{2}$. It is easy to show that if $f(x) = kx^a$, with $a, k > 0$, this ratio is constant;

$$\frac{V(r)}{C(r)} = \frac{2}{a+2}.$$

We were surprised to discover a sort of converse: The only twice differentiable increasing functions for which the ratio $V(r)/C(r)$ is constant are power functions $f(x) = kx^a$, with $a, k > 0$. Showing that this geometric property characterizes the two-parameter family of power functions provides a nice application of elementary differential equations.

First, if $f(x) = kx^a$, with $a, k > 0$, then using the method of cylindrical shells, we see that

$$\begin{aligned} V(r) &= 2\pi \int_0^r x f(x)\, dx = 2\pi k \int_0^r x^{a+1}\, dx \\ &= \frac{2\pi k r^{a+2}}{a+2}. \end{aligned}$$

Hence,

$$\frac{V(r)}{C(r)} = \frac{2\pi k r^{a+2}}{a+2} \frac{1}{\pi r^2 k r^a} = \frac{2}{a+2}.$$

Proposition. Suppose f is a positive, strictly increasing, twice differentiable function on an interval $I = (0, b)$, and the ratio $V(r)/C(r)$ is constant for $r \in I$. Then $f(x) = kx^a$, for some positive constants k and a.

Proof. By differentiating the constant quotient $V(r)/C(r)$ we see that $C(r)V'(r) - C'(r)V(r) = 0$. Substituting $C(r) = \pi r^2 f(r)$ and applying the fundamental theorem of calculus to find $V'(r)$, we get

$$\left[\pi r^2 f(r)\right]\left[2\pi r f(r)\right] - \left[\pi r^2 f'(r) + 2\pi r f(r)\right]\left[2\pi \int_0^r x f(x)\, dx\right] = 0,$$

or

$$\int_0^r x f(x)\, dx = \frac{[r f(r)]^2}{r f'(r) + 2 f(r)}. \tag{1}$$

Note that our hypotheses imply that r, $f(r)$, and $f'(r)$ are all positive on I, so the denominator on the right side of (1) is nonzero.

Differentiating (1) gives

$$rf = \frac{(rf' + 2f)2rf(rf' + f) - (rf)^2(rf'' + 3f')}{(rf' + 2f)^2},$$

where for simplicity we have suppressed the argument r of the functions. After a bit of algebraic manipulation, this equation becomes

$$rf'^2 - ff' - rff'' = 0 \tag{2}$$

Changing to the customary notation of differential equations, we will let x represent the independent variable (instead of r) and let $y = f(x)$ denote the dependent variable, so that (2) becomes

$$xy'^2 - yy' - xyy'' = 0. \tag{3}$$

Since x, y, and y' are all positive on I, we may divide (3) by $-xyy'$ to get

$$\frac{y''}{y'} - \frac{y'}{y} = -\frac{1}{x}. \tag{4}$$

Integrating yields

$$\ln y' - \ln y = -\ln x + c,$$

or

$$\ln \frac{y'}{y} = c - \ln x.$$

Applying the exponential function to both sides then gives

$$\frac{y'}{y} = \frac{a}{x},$$

where $a = e^c$ is a positive constant.

Integrating again, we conclude that $\ln y = a \ln x + K$, hence $y = kx^a$, where $k = e^K$ is a positive constant.

———— o ————