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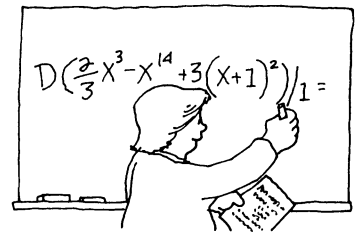

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Rectangular-to-Polar Folding Fans

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Here is a project for students who are studying polar coordinates, whether in a calculus, precalculus, or trigonometry course. Students learn and perform a paper folding technique that transforms a sketch of the graph of $y = f(x)$ in rectangular coordinates, for $0 \le x \le 2\pi$, into a sketch of the graph of the same function in polar coordinates over the interval $0 \le \theta \le 2\pi$. For example, if they sketch $y = 1 + \sin x$, then they can produce a fan upon which appears a sketch of the cardioid $r = 1 + \sin \theta$ by following the paper folding process illustrated in the accompanying figure. The instructions and comments that follow will be easier to digest if the reader first studies that figure long enough to glean the general concept behind the folding, referring back to the figure later as needed.

Besides being fun, the project can serve other purposes. One can give a computer assignment requiring a student to display and print out a suitably chosen window view of a function (although hand drawn sketches will do almost as nicely). More importantly, the project should enhance a student's understanding of geometric connections between the shape of a given curve $y = f(x)$ in rectangular coordinates over $0 \le x \le 2\pi$ (satisfying $f(x) \ge 0$) and the shape of the corresponding polar graph $r = f(\theta)$.

What exactly is the connection? Each point $(x, y) = (a, f(a))$ on $y = f(x)$ naturally corresponds to the plotted point $(r, \theta) = (f(a), a)$ on $r = f(\theta)$. But while the point $(x, y) = (a, f(a))$ is on the $x = a$ and at distance $f(a)$ above the x-axis, the point $(r, \theta) = (f(a), a)$ is on the terminal side of the angle $\theta = a$ and at distance $f(a)$ from the origin. Thus the vertical half-line $\{(a, y): y \ge 0\}$ in rectangular coordinates naturally corresponds by rigid motion to the ray $\{(r, a): r \ge 0\}$ in polar coordinates. The fan folding process merely repositions each half-line $\{(a, y): y \ge 0\}$ to the position of the polar ray $\{(r, a): r \ge 0\}$, dragging along with it the point $(x, y) = (a, f(a))$ to the polar location $(r, \theta) = (f(a), a)$. Or, less formally, the points $(a, 0)$ of these half-lines $\{(a, y): y \ge 0\}$ get merged to a single point at the origin, with the joined rays now fanned out evenly. But since the polar angle increases in the counterclockwise sense while the x-coordinate increases from left to right, we turn over our sketch of $y = f(x)$ before fanning out its rays to get a sketch of $r = f(\theta)$.

A class given this project would of course benefit from first seeing a physical example of the final product, a working fan, or an unfinished fan in one or more of its stages of construction. Be forewarned that, unless you supply detailed instructions such as those given here, all too many students will find ways to mess up the procedure for constructing the fans.

Tools required. Scissors, three popsicle sticks, stapler, rubber band, ruler, curve sketching software, and computer printing facilities.

Sample instructions. Make a fan of some interesting polar curve $r=f(\theta)$. To do this, first select an appropriate function for f. Unless you are particularly clever, choose a continuous f such that $f(\theta)\geq 0$ for all θ in the interval $0\leq\theta\leq 2\pi$. Next, find software and computing facilities that allow you to print out a sketch of $y=f(x)$ in rectangular coordinates viewed over any specified rectangular window $a\leq x\leq b$, $c\leq y\leq d$. The printed graph should be at least 5 inches wide. Most curve sketching software will conform to these requirements. Bring a ruler to the computer facilities.

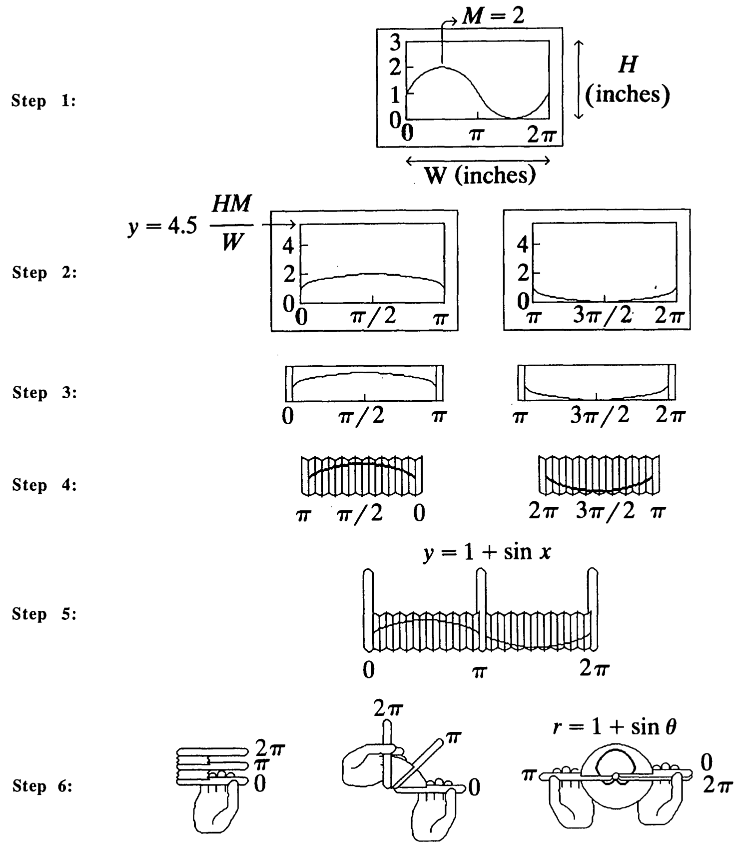
Step 1. Have the computer sketch $y=f(x)$, graphed in rectangular coordinates over the interval $0\leq x\leq 2\pi$. Estimate the largest output, M, attained by $f(x)$ on that interval (i.e., estimate the absolute maximum value), even if this means resketching several times using different vertical ranges $c\leq y\leq d$ until M is apparent. With a ruler, measure the width W and the height H occupied by the sketch's window including any blank space where no points are plotted. Do not bother to print out this display.

Step 2. Have the computer sketch the curve $y=f(x)$ over just the window $0\leq x\leq\pi$, $0\leq y\leq 4.5\ HM/W$, and print out the sketch. (You might want to print out a second copy in case you mess up later.) Then do the same over the window $\pi\leq x\leq 2\pi$, $0\leq y\leq 4.5\,HM/W$. If things have gone well each window on your printouts should be about four or five times as wide as the altitude above the x-axis of a point plotted at y-coordinate M. You are now done with the computer.

Step 3. Cut out a rectangular strip from each printout ($0\leq x\leq\pi$ and $\pi\leq x\leq 2\pi$), leaving a little room as follows. The x-axis should be along the bottom of each strip, and the top of each strip should be just above y-coordinate M, so that all of the curve appears within the strip. Cut the left and right sides of the strip just a bit beyond the ends of the plotted intervals (about $\frac{3}{8}''$ margins), leaving room for stapling the strips to the popsicle sticks.

Step 4. On the *back* side of your strips, retrace the computer-plotted curve. (Try holding the printout up to a window or bright light to do the tracing.) A narrow felt tip marker works nicely. Then fold each strip into a fan using roughly equal "Δx" widths for the folds. Folds of width $\frac{3}{8}''$ work well, since that's about the width of a popsicle stick.

Step 5. Along the margins, connect the front sides of the two strips into a single sketch of $y=f(x)$ over the entire interval $0\leq x\leq 2\pi$ by stapling them, with one end of the sticks just touching (or slightly above) the x-axis. Then staple a stick at



each of the other two strip ends, again with the stick ends just slightly above the x-axis. You now have vertical sticks along the lines $x = 0$, $x = \pi$, and $x = 2\pi$.

Step 6. Fold up your fan, bringing all three sticks together, then turn it around so that the back side faces you. Hold it horizontally with the paper end on the left, so that the other ends of the sticks point in the polar direction $\theta = 0$. Gently test your fan with your left hand by opening it a full 360° counterclockwise (you might wish you had a third hand!)—or simply unfold the fan on a flat surface. If you already know what the polar curve $r = f(\theta)$ looks like, you should recognize what you see. If your fan will not open up a full 360°, perhaps you did not trim just above y-coordinate M in step 3. If all is well, fold your fan back together and secure it with a rubber band. Before turning in your completed work, write your name on one stick, along with the formula for your function $r = f(\theta)$.

Since printers and curve sketching software are not all alike, in the above instructions I tried to allow for such technological variations. If your class's printing facilities and software are uniform, you can simplify these instructions. The objective is to get students to print out sketches leading to the creation of a fan, formed from two folding strips whose height-to-width ratio is roughly 1 to 4. Some classes may be able to work things out for themselves based on that objective alone, but the risk is that several disappointed students will produce nonfunctioning fans.

Fanning out. Here are some suggestions and alternatives, their usefulness depending on the abilities of the students involved.

If you leave it to the student to choose f (as in the sample instructions), do not waste many interesting examples such as $r = 1 + \sin\theta$ by showing them as finished fans. The ellipse $r = 6/(2 + \cos\theta)$ is a good example to show. The circle (or one-leafed rose) $r = \sin\theta$ and the four-leaved rose $r = \sin 2\theta$ (which produces only two leaves here) nicely show the result of deleting all points with negative radii.

You might consider how to use "overlay" features to handle the possibility that $f(\theta) < 0$, for which an overlay of $r = -f(\theta - \pi)$ is often successful. Or, if more than just the interval $0 \le \theta \le 2\pi$ is involved in a polar sketch, overlays of $r = f(\theta - 2\pi)$, $r = f(\theta + 2\pi)$, $r = f(\theta - 4\pi)$, and so on, may be needed.

Ask students to explain why the choice $0 \le y \le 4.5\ HM/W$ was made, and have them find the least positive constant c such that $0 \le y \le c\ HM/W$ might produce a fully functional folding fan (Answer: $c = \pi$). This is a fairly difficult question for most students, but hardly impossible.

Tell students who are studying or have studied multivariate calculus to shade in two rectangles, $\pi/4$ wide by $M/4$ high, on the unfolded strips: one at the top of the strip and one near the middle. Have them use the resulting fan to help explain the r factor in the $dx\,dy \to r\,dr\,d\theta$ change-of-variable formula for multiple integrals.

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A Sequence Related to the Harmonic Series

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Since the harmonic series diverges, the series $\sum_{k=n}^{\infty} 1/k$ also diverges for each positive integer n. It follows that there is a least positive integer $a(n)$ such that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{a(n)} > 1,$$

and the resulting sequence $a(n)$ is the focus of our study. Inductions clearly show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} < 1 \qquad \text{and}$$

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n-2} > 1 \qquad \text{for } n \ge 2,$$

so we see that $2n - 1 < a(n) \le 3n - 2$ for all $n \ge 2$.

These bounds on $a(n)$ suggest that the sequence $a(n)/n$ may converge to a limit and, since $\int_n^{ne} 1/x\,dx = 1$, we expect that limit to be the number e.

To verify this limit, first observe from the definition of $a(n)$ that

$$1 < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{a(n)} \le 1 + \frac{1}{a(n)}.$$