Locating Multiples of Primes in Pascal's Triangle

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In this capsule, we show how to find all the binomial coefficients that are multiples of any given prime. Our arguments are essentially visual, illustrating how all such multiples can be located in triangular blocks within Pascal's triangle.

The key observation is that when several consecutive entries on one row of Pascal's triangle are all multiples of a given number, then (because the sum of two multiples of a number is again a multiple of the same number) an entire triangular section of binomial coefficients consists of such multiples.

To illustrate these triangular blocks, Figure 1 shows a portion of Pascal's triangle with the complete 11th row, R_{11}. Observe that every interior entry on R_{11} is a multiple of 11, giving rise to a $10 \times 10 \times 10$ block of multiples of 11, from the 11th to the 20th rows. Similarly, all interior entries on row R_{13} are multiples of 13, and the first six (as well as the last six) consecutive entries on row R_{14} are multiples of 7. The overlap (intersection) of three triangular sections is a block consisting of multiples of $7 \cdot 11 \cdot 13 = 1001$.

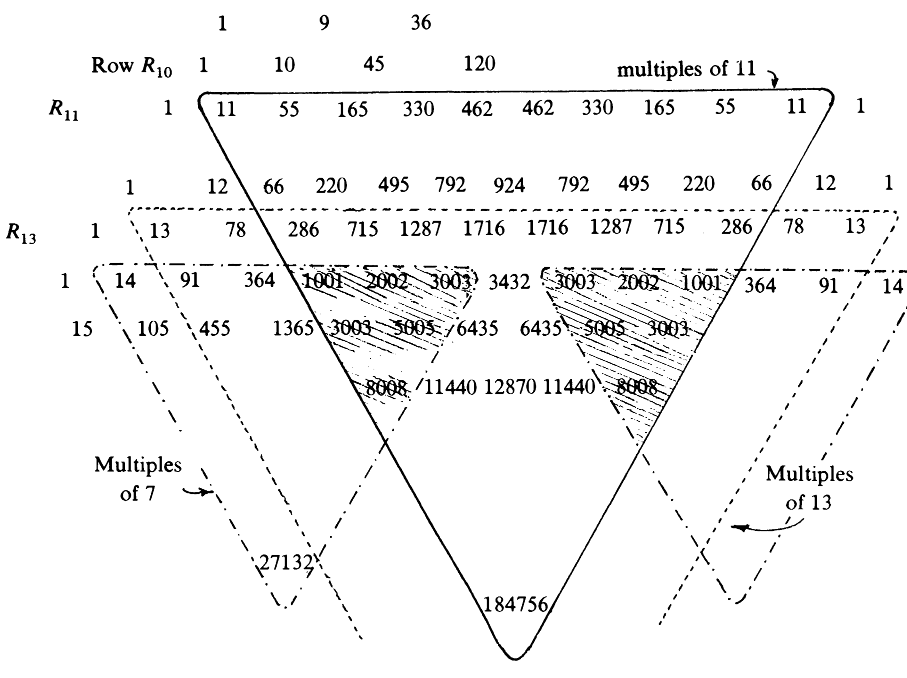

Figure 1. Shaded triangles consist of multiples of $7 \cdot 11 \cdot 13 = 1001$.

In order to proceed, we need the following useful result.

Property (∗). *An integer n ($n > 1$) is prime if and only if n divides every interior entry in the nth row of Pascal's triangle.*

Verification of Property (∗) is straightforward. If n is prime, all the entries on the nth row of Pascal's triangle (except the exterior 1's) are multiples of n. This follows from the definition $\binom{n}{r} = n!/r!(n-r)!$, since no factor of the denominator can divide n when n is prime. Suppose, conversely, that n is not prime; say $n = kp^r$, where p is prime and relatively prime to k. We show that $\binom{n}{p}$ is not divisible by n. After dividing p from the numerator and denominator of $\binom{n}{p}$, the numerator would have the form

$$kp^{r-1}(n-1)(n-2)\cdots(n-p+1).$$

If $\binom{n}{p}$ had a factor of n, then the numerator would have to have another factor of p, but this is impossible since each of the factors after the first is relatively prime to p.

To find all the multiples of a given prime p in Pascal's triangle, it is convenient to consider only remainders modulo p. Then the binomial coefficients that contain p as a factor are precisely the *zero remainders modulo p*. Thus, by Property (*), the pth row R_p consists of two exterior 1's separated by $p-1$ consecutive zeros. Since no prime p is the product of two smaller factors, *no binomial coefficient above the pth row can be a multiple of p*. In other words, the triangular portion T_p of Pascal's triangle above the pth row consists entirely of nonzero remainders. See Figure 2.

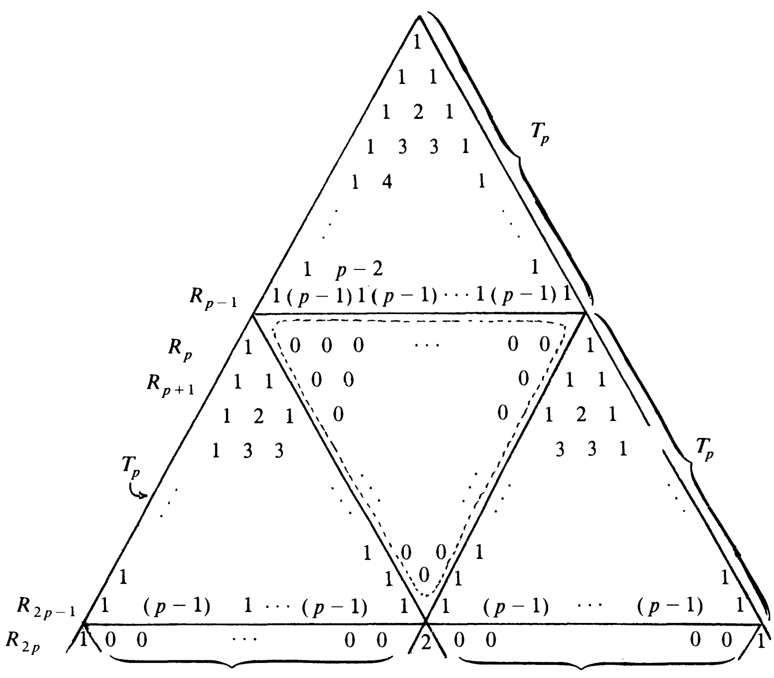

Figure 2. Pascal's Triangle Modulo p.

From our knowledge of the string of zeros on the pth row, we know that the preceding row R_{p-1} consists of alternating entries of 1 and $p-1$, and the following row R_{p+1} has two 1's at each end separated by $p-2$ zeros. In fact, the string of $p-1$ zeros on R_p gives rise to a whole triangular array of zeros (enclosed in dotted lines in Figure 2). Since the triangular block of zeros is bordered on the lower edges by 1's, the trapezoid with bases R_p and R_{2p-1} contains two copies of T_p. And R_{2p} has two sequences of $p-1$ zeros separated by a single remainder 2.

As we continue downward from row R_{2p} the two zero-sequences give rise to three triangular nonzero-blocks: two are duplicates of T_p and one is the block $2T_p$ whose entries are precisely twice (mod p) the corresponding entries in T_p. From these three blocks, $T_p, 2T_p, T_p$, we see that the interior entries of row R_{3p} consist of three sequences of $p-1$ zeros, each separated by the number 3. Figure 3 illustrates this for $p=5$.

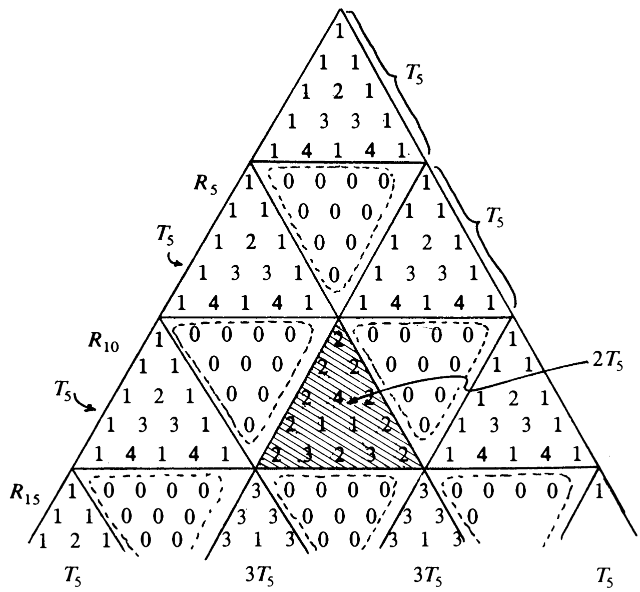

Figure 3. Pascal's Triangle Modulo 5.

From Figure 4, the structure of the whole of Pascal's triangle (mod 5) should become apparent; the pattern is the same for any prime. The multiples of 5 occur in triangular blocks of size $4 \times 4 \times 4$ down to row R_{25}, which has 24 consecutive multiples of 5 separating the exterior 1's. This happens because the same multiples of T_5 occur in the same locations relative to the large block T_{25}, as do the multiples of 1 in T_5; that is, exactly the same structure is repeated in T_{25} as in T_5, with the 1's replaced by T_5. Then, on a still larger scale, T_{25} and its multiples replace T_5, leading to another full row of interior multiples of 5 on row R_{125}, and the structure is inflated again by another factor of 5, with T_{25} replacing T_5.

More generally, for any prime p, the multiples of p occur in blocks of size $(p-1) \times (p-1) \times (p-1)$, down to row R_{p^2}, where p^2-1 consecutive multiples of p give rise to a block of

$$(p^2-1)+(p^2-2)+\cdots+1=\frac{(p^2-1)p^2}{2}$$

multiples of p. The repetition continues, duplicating larger and larger blocks, as in Figure 5.

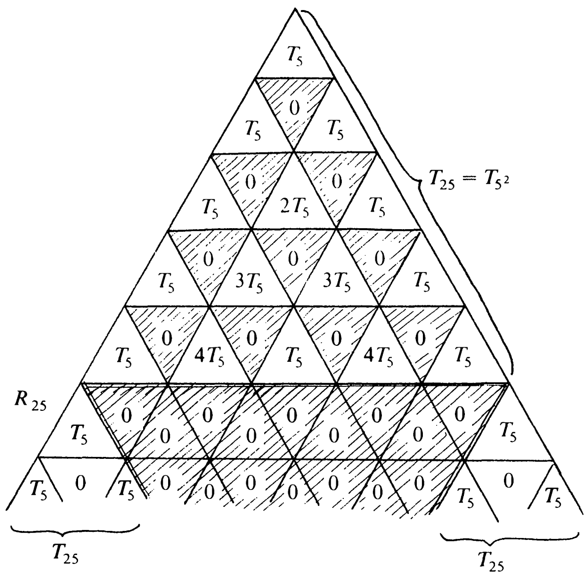

Figure 4. Pascal's Triangle Modulo 5.

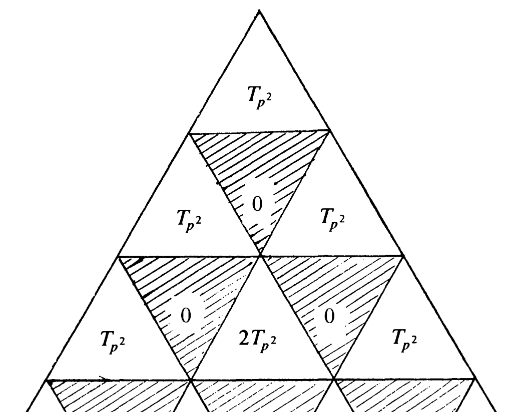

Figure 5. Pascal's Triangle Modulo p.

If we use the same modulus throughout, we may think of "stepping back" to view more and more of the same triangle. In Figure 3, we see individual remainders identified in blocks; Figure 4 shows the blocks as elements, grouped into blocks T_{p^2}; and Figure 5 shows these $p^2 \times p^2 \times p^2$ sized blocks and their multiples, which together will form $p^3 \times p^3 \times p^3$ sized blocks, and so on.

Summing up, we conclude that for a given prime p, the binomial coefficients that are multiples of p are all contained in triangular blocks each of whose upper row is part of some row R_{kp}. There are no multiples of p above row R_p. All the interior entries on a row are multiples of p on precisely those rows that are powers of p.

Below each row R_{p^n}, there are repetitions of the structure (mod p) of the portion of the triangle above that row.

Applications. We can readily locate the first occurrence (in Pascal's triangle) of a multiple of a given product of distinct primes. The smallest multiple of a number such as $2001 = 3 \cdot 23 \cdot 29$ is, of course, $\binom{2001}{1} = 2001$; but that doesn't appear in the triangle until row R_{2001}. The first time 2001 occurs as a factor is on row R_{29}. To find multiples of 2001 in the triangle, we need to find overlapping blocks of multiples of 3, multiples of 23, and multiples of 29. There is no multiple of 29 before row R_{29}, so we have only to determine where the other factors are located. Beginning with row R_{23}, we have a triangular block of multiples of 23 which extends downward for twenty-two rows and overlaps part of R_{29}. Since $27 = 3^3$, all interior entries on row R_{27} are multiples of 3. Therefore, each binomial coefficient in the shaded triangle T in Figure 6 is a multiple of 2001. The smallest such entry is $\binom{29}{7} = 1560780 = (2001)(780)$; the largest entry in T is $\binom{44}{22} = (2001)(1051523720)$. Altogether there are $16 + 15 + \cdots + 2 + 1 = 136$ multiples of 2001 in T.

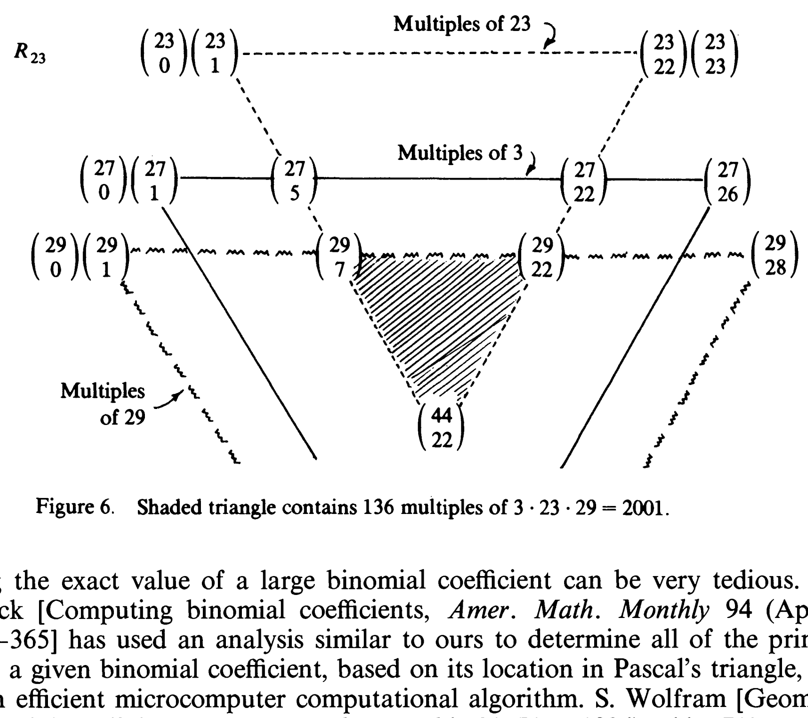

Figure 6. **Shaded triangle contains 136 multiples of $3 \cdot 23 \cdot 29 = 2001$.**

Finding the exact value of a large binomial coefficient can be very tedious. P. Goetgheluck [Computing binomial coefficients, *Amer. Math. Monthly* 94 (April 1987) 360–365] has used an analysis similar to ours to determine all of the prime divisors of a given binomial coefficient, based on its location in Pascal's triangle, to develop an efficient microcomputer computational algorithm. S. Wolfram [Geometry of binomial coefficients, *Amer. Math. Monthly* 91 (Nov 1984) 566–571] examined the patterns of divisibility of binomial coefficients to determine the fractal dimension of the patterns. I am also indebted to the referees for pointing out the additional articles by C. T. Long [Pascal's triangle modulo p, *Fib. Quart. J.* 19 (1981) 458–463] and D. Singmaster ["Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers" in *A Collection of Manuscripts Related to the Fibonacci Sequence*, 98–113, The Fibonacci Association, Santa Clara, CA, 1980].