Assume that (*) holds for each $i \in S(n)$. Then

$$p\left((n+1)^2 - 1\right) = \left[a\left((n+1)^2-1\right) + b\left((n+1)^2-1\right)\right]/2$$

$$= 2^{-1} + 2^{-4} + \cdots + 2^{-n^2} + 2^{-(n+1)^2},$$

which is strictly less than $p$. Accordingly,

$$a\left((n+1)^2\right) = p\left((n+1)^2-1\right) = 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2}$$

and

$$b\left((n+1)^2\right) = b\left((n+1)^2-1\right) = 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2+1}$$

$$= 2^{-1} + \cdots + 2^{-(n+1)^2} + 2^{-(n+1)^2}.$$

This shows that $[a(i), b(i)]$ satisfies (*) for $i = (n+1)^2$ in $S(n+1)$. Suppose $[a(i), b(i)]$ satisfies (*) for $i = (n+1)^2 + k$ $(0 < k < 2n+2)$ in $S(n+1)$. Then

$$p\left((n+1)^2 + k\right) = \left[a\left((n+1)^2+k\right) + b\left((n+1)^2+k\right)\right]/2$$

$$= 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2} + 2^{-[(n+1)^2+k+1]},$$

which is easily seen to be larger than $p$. Thus,

$$a\left((n+1)^2+k+1\right) = a\left((n+1)^2+k\right) = 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2}$$

and

$$b\left((n+1)^2+k+1\right) = p\left((n+1)^2+k\right) = 2^{-1} + 2^{-4} + \cdots + 2^{-[(n+1)^2+k+1]},$$

so that $[a(i), b(i)]$ satisfies (*) for $i = (n+1)^2 + k + 1$ in $S(n+1)$. In particular, (*) holds for each $i \in S(n+1)$, and this concludes our induction proof of (*).

Now, using (*) to calculate $p(i)$ for $i = n^2 \in S(n)$ and $p(i)$ for $i = n^2 - 1 \in S(n-1)$, we find

$$\frac{|p(n^2) - p|}{|p(n^2-1) - p|} = \frac{2^{2n}(1 - 2^{-2n} - 2^{-4n-3} - 2^{-6n-8} - \cdots)}{1 + 2^{-2n-3} + 2^{-4n-8} + 2^{-6n-15} + \cdots}.$$

Since this ratio is unbounded for increasing $n$, the sequence $p(i)$ cannot be linearly convergent!

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## Nested Polynomials and Efficient Exponentiation Algorithms for Calculators

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Lyle Cook and James McWilliams [TYCMJ 14 (January 1983) 52–54] presented a simple algorithm for approximating the cube root of a number on a calculator equipped with a square root key but with no general power or root key. This stimulated further thinking on how to combine some elementary notions—binary representations and the nested form for polynomials—to produce efficient exponentiation algorithms for calculators with square and square root keys.

The simplest case of exponentiation involves raising a number to a positive integer power. For example, $(1.23)^{18}$ can be obtained by repeated multiplication of 1.23. But this can be improved by judiciously using the $\boxed{x^2}$ key. Once 1.23 is stored in memory and entered in the display, each of the following sequences works:

$$\boxed{x^2}\ \boxed{x^2}\ \boxed{x^2}\ \boxed{x^2}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=} \tag{1}$$

$$\boxed{x^2}\ \boxed{x^2}\ \boxed{x^2}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}\ \boxed{x^2} \tag{2}$$

To seek a systematic procedure for finding the most efficient sequence of steps, first consider the nested form of the binary representation of 18:

$$\begin{aligned}18 &= 1\cdot 2^4 + 0\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\\ &= \{1\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2 + 1\}\cdot 2 + 0\\ &= \{[1\cdot 2^2 + 0\cdot 2 + 0]\cdot 2 + 1\}\cdot 2 + 0\\ &= \{[(1\cdot 2 + 0)\cdot 2 + 0]\cdot 2 + 1\}\cdot 2 + 0\\ &= (1\cdot 2\cdot 2\cdot 2 + 1)\cdot 2.\end{aligned}$$

Therefore,

$$\begin{aligned}(1.23)^{18} &= (1.23)^{(1\cdot 2\cdot 2\cdot 2+1)\cdot 2}\\ &= \left\{\left[\left\{(1.23)^2\right\}^2\right]^2 \times (1.23)\right\}^2,\end{aligned}$$

which corresponds to key sequence (2).

In general, the base $b$ representation of a positive integer $n$,

$$n = a_0 b^k + a_1 b^{k-1} + \cdots + a_{k-1} b + a_k,$$

has nested from

$$n = (\cdots(((a_0)\cdot b + a_1)\cdot b + a_2)\cdot b + \cdots + a_{k-1})\cdot b + a_k.$$

This nested form of a number's representation is efficient for evaluation because it requires only addition and multiplication. For the particular case of a binary representation, our evaluation requires only multiplications by 2 and additions of 1. Thus, when an exponent $n$ is expressed in nested binary form, evaluation of $x^n$ requires only squarings and multiplications by the base $x$. Since $n = 1a_1a_2\cdots a_{k-1}a_k$ in base 2, the coefficient $a_0$ of $2^k$ will always be 1; we can therefore ignore this leading 1 when considering $x^n$. Note, furthermore, that for each coefficient $a_i = 1$, it is necessary to square and multiply by the base $x$; for each coefficient $a_i = 0$, it is necessary only to square. For example, since 13 is 1101 in base 2, we obtain $x^{13}$ by

$$\underbrace{\boxed{x^2}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{x^2}}_{0}\ \underbrace{\boxed{x^2}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1} \tag{3}$$

To illustrate our approach for fractional exponents, consider extracting cube roots as discussed by Cook and McWilliams. The binary representation

$$\tfrac{1}{3} = 0 + 0 \cdot (\tfrac{1}{2}) + 1 \cdot (\tfrac{1}{2})^2 + 0 \cdot (\tfrac{1}{2})^3 + 1 \cdot (\tfrac{1}{2})^4 + \cdots$$

can be written

$$\tfrac{1}{3} = 0.010101 \cdots .$$

Let's look at the first five digits, 0.0101. These are the coefficients *read off in reverse order* of a polynomial in powers of $\frac{1}{2}$ which approximates $\frac{1}{3}$. Specifically,

$$\begin{aligned}\tfrac{1}{3} &\doteq 1 \cdot (\tfrac{1}{2})^4 + 0 \cdot (\tfrac{1}{2})^3 + 1 \cdot (\tfrac{1}{2})^2 + 0 \cdot (\tfrac{1}{2}) + 0 \\ &= \left\{\left[(1 \cdot \tfrac{1}{2} + 0) \cdot \tfrac{1}{2} + 1\right] \cdot \tfrac{1}{2} + 0\right\} \cdot \tfrac{1}{2} + 0 \\ &= (\tfrac{1}{2} \cdot \tfrac{1}{2} + 1) \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} .\end{aligned}$$

Thus, the corresponding approximation for $x^{1/3}$ ($\doteq x^{(\frac{1}{2}\cdot\frac{1}{2}+1)\cdot\frac{1}{2}\cdot\frac{1}{2}}$) is obtained by the key sequence

$$\boxed{\sqrt{x}}\ \boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}\ \boxed{\sqrt{x}}\ \boxed{\sqrt{x}} \tag{4}$$

Now we see that our original procedure applies, with $\boxed{x^2}$ replaced by $\boxed{\sqrt{x}}$, when considering $\frac{1}{3} = 10100$ in base $b = \frac{1}{2}$. Similarly, if 0.010101 is used, the resulting key sequence for approximating $x^{1/3}$ is

$$\underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}}_{0}$$

Evidently, the pattern may be continued through as many cycles as desired, corresponding to including ever more terms in the binary expansion of $\frac{1}{3}$. This is the algorithm presented by Cook and McWilliams.

For the general case of a proper fraction $p/q$, the binary representation must terminate or repeat. In either case, our method may be used to approximate $x^{p/q}$ for a given base $x$. By way of illustration, consider $2^{3/5}$. Our approach will be to first obtain $x = 2^{1/5}$ and then (since 3 has a binary representation of 11) use $x^3 = \boxed{x^2}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}$ . We begin by noting that $\frac{1}{5}$ has a repeating binary expansion $\overline{0.0011}$.

For a first approximation, $\frac{1}{5} \doteq 0.0011$, the key sequence for $x^{1/5}$ determined by 11000 (the binary coefficients in reverse order) is

$$\underbrace{\boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}}_{0} .$$

Using two cycles in our approximation, $\frac{1}{5} = 0.00110011$, the key sequence (as determined by 11001100) is

$$\underbrace{\boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{\sqrt{x}}\ \boxed{\times}\ \boxed{\text{Recall}}\ \boxed{=}}_{1}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}}_{0}\ \underbrace{\boxed{\sqrt{x}}}_{0} .$$

We can proceed in this manner by ignoring the first 1, and replacing each following 1 by $\boxed{\sqrt{x}}$ $\boxed{\times}$ $\boxed{\text{Recall}}$ $\boxed{=}$ and each 0 by $\boxed{\sqrt{x}}$.

The table below lists approximations for $2^{1/5}$ and $2^{3/5} = (2^{1/5})^2 \times 2^{1/5}$ corresponding to $k$ cycles ($k = 1, 2, \ldots, 7$) in the binary approximation of $\frac{1}{5}$.

| $k$ | Approximate value of $2^{1/5}$ | Approximate value of $2^{3/5}$ |
|---|---|---|
| 1 | 1.1387886 | 1.4768260 |
| 2 | 1.1480765 | 1.5132563 |
| 3 | 1.1486595 | 1.5155628 |
| 4 | 1.1486959 | 1.5157068 |
| 5 | 1.1486982 | 1.5157160 |
| 6 | 1.1486983 | 1.5157163 |
| 7 | 1.1486984 | 1.5157167 |

The last approximations of $2^{1/5}$ and $2^{3/5}$ each agree with their correct values in the first six decimal places.

## Please Feed the Archives by *R. P. Boas*

It is saddening to dream on
What was lost when Bernie Riemann
Used his letters from Cauchy to light a fire.

If Weyl had left a note
On what Noether never wrote,
Higher algebra might be a good deal higher.

If you want to stake a claim
On everlasting fame
Then cherish every single bit of writing,

For your discards won't come back,
And they generate a lack
Of information, that the future will be fighting.

If you just will act with prudence
Some future thesis students
Can one day clarify a little mystery;

For what you would have thrown
Away, when it is known,
Will assure you of a footnote in their history.

Though the letters in your "files"
May be just untidy piles,
They'll throw light on all your failures and successes,

So please do nothing rash
With your daily office trash,
But ship it to the archives down in Texas.

Never throw the stuff away,
Just send it—come what may—
To the celebrated Archives down in Texas!