CLASSROOM CAPSULES

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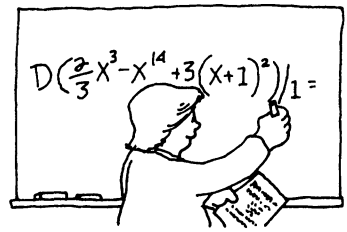

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Natural Logarithms via Long Division

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Many calculus students are aware of the beautiful series expansion for $\ln 2$. It is the purpose of this note to develop such expansions for $\ln 3$ and $\ln 4$ by elementary methods. The basic ideas are: $\frac{1}{1+x}$, $\frac{1+2x}{1+x+x^2}$, $\frac{1+2x+3x^2}{1+x+x^2+x^3}$, etc., have the form "$\frac{du}{u}$", and, long division. For another approach see **[1]**. We begin with the familiar, $\ln 2$.

$$\begin{aligned}\ln 2 &= \int_0^1 \frac{1}{1+x}\,dx \\ &= \int_0^1 \left[(1-x)+(x^2-x^3)+\cdots+(x^{2k-2}-x^{2k-1})+\frac{x^{2k}}{1+x}\right]dx \\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\int_0^1 \frac{x^{2k}}{1+x}\,dx.\end{aligned}$$

Using partial sum notation ($s_n = a_1 + a_2 + \cdots + a_n$), we have shown

$$|\ln 2 - s_{2k}| = \left|\int_0^1 \frac{x^{2k}}{1+x}\,dx\right| < \frac{1}{2k+1}.$$

But then

$$\begin{aligned}|\ln 2 - s_{2k+1}| &\le |\ln 2 - s_{2k}| + |s_{2k} - s_{2k+1}| \\ &< \frac{1}{2k+1} + \frac{1}{2k+1} = \frac{2}{2k+1}\end{aligned}$$

Because $n = 2k$ or $2k+1$,

$$|\ln 2 - s_n| < \frac{2}{n}$$

and

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

We mimic this argument for $\ln 3$.

$$\begin{aligned}
\ln 3 &= \int_0^1 \frac{1+2x}{1+x+x^2}\,dx \\
&= \int_0^1 \Bigg[(1 + x - 2x^2) + (x^3 + x^4 - 2x^5) + \cdots \\
&\qquad + (x^{3k-3} + x^{3k-2} - 2x^{3k-1}) + \frac{x^{3k} + 2x^{3k+1}}{1+x+x^2} \Bigg] dx \\
&= \left(1 + \frac{1}{2} - \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} - \frac{2}{6}\right) + \cdots \\
&\qquad + \left(\frac{1}{3k-2} + \frac{1}{3k-1} - \frac{2}{3k}\right) + \int_0^1 \frac{x^{3k} + 2x^{3k+1}}{1+x+x^2}\,dx.
\end{aligned}$$

Using partial sums,

$$|\ln 3 - s_{3k}| = \left| \int_0^1 \frac{x^{3k} + 2x^{3k+1}}{1+x+x^2}\,dx \right| < \frac{3}{3k+1}.$$

But then

$$\begin{aligned}
|\ln 3 - s_{3k+1}| &\le |\ln 3 - s_{3k}| + |s_{3k} - s_{3k+1}| \\
&< \frac{3}{3k+1} + \frac{1}{3k+1} = \frac{4}{3k+1},
\end{aligned}$$

and

$$\begin{aligned}
|\ln 3 - s_{3k+2}| &\le |\ln 3 - s_{3k}| + |s_{3k} - s_{3k+1}| + |s_{3k+1} - s_{3k+2}| \\
&< \frac{3}{3k+1} + \frac{1}{3k+1} + \frac{1}{3k+2} < \frac{5}{3k+1}.
\end{aligned}$$

Since $n = 3k$, or $3k+1$, or $3k+2$,

$$|\ln 3 - s_n| < \frac{5}{n-1}, \qquad n \ge 2.$$

Thus,

$$\ln 3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \cdots .$$

The reader may similarly show

$$\begin{aligned}\ln 4 &= \int_0^1 \frac{1+2x+3x^2}{1+x+x^2+x^3}\,dx\\ &= \left(1+\frac{1}{2}+\frac{1}{3}-\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{3}{8}\right)+\cdots\\ &\quad+\left(\frac{1}{4k-3}+\frac{1}{4k-2}+\frac{1}{4k-1}-\frac{3}{4k}\right)\\ &\quad+\int_0^1 \frac{x^{4k}+2x^{4k+1}+3x^{4k+2}}{1+x+x^2+x^3}\,dx\\ &= 1+\frac{1}{2}+\frac{1}{3}-\frac{3}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{3}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}-\frac{3}{12}+\cdots.\end{aligned}$$

it is straightforward though cumbersome to show

$$\begin{aligned}\ln N &= 1+\frac{1}{2}+\cdots+\frac{1}{N-1}-\frac{N-1}{N}+\frac{1}{N+1}+\frac{1}{N+2}+\cdots\\ &\quad+\frac{1}{2N-1}-\frac{N-1}{2N}+\cdots.\end{aligned}$$

References

1. C. Kicey and S. Goel, A series for ln k, *American Mathematical Monthly* 105 (1998).

Simple Geometric Solutions to De l'Hospital's Pulley Problem

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In a recent paper, Hahn [2] discusses De l'Hospital's solution to the following problem. A weight is attached by a cord to a point in the ceiling. The cord runs over a pulley attached by a cord of length r to a point in the ceiling at distance d from the first as shown in Figure 1. The problem is to find the equilibrium configuration,

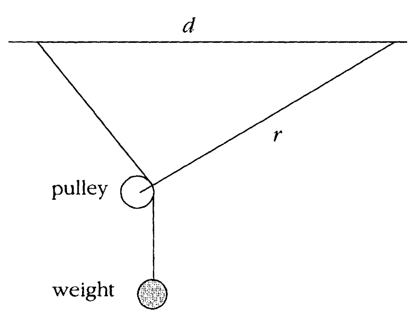

Figure 1. Arrangement of weight and pulley.