CLASSROOM CAPSULES

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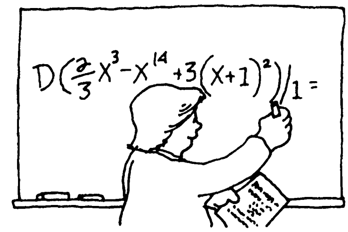

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Normal Lines and the Evolute Curve

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In almost every calculus book can be found problems of the form "Given a point (x_0, y_0) find an equation of the normal line to the graph of $y=f(x)$ passing through the point." This leads to the more interesting question of how many normal lines of a given smooth curve pass through a given point not on the curve, and we believe this question provides a source of good exercises or projects for an introductory calculus class. The analysis can also provide opportunities for creative computer graphics.

Let C be a smooth curve in the xy-plane i.e., the graph of a differentiable function, hence having a normal line at each point. Let (a, b) be any point. We define the *N-rank* of (a, b) to be the number of points (x, y) on C whose normal line contains the point (a, b). We note that distinct points on C may have the same normal line, and our definition of the N-rank of (a, b) counts the number of normal lines, with multiplicities, that contain (a, b). For example the cubic

$$x = t, \qquad y = \frac{c^2}{9}t^3 + ct^2 + t$$

has the line $y = -x$ normal at both the points $(0, 0)$ and $(-6/c, 6/c)$ on its graph.

Our problem is to describe graphically those regions in the plane having N-rank n, $n = 0, 1, 2, \ldots$. We will give a complete solution when C is the graph of a polynomial function.

For example,

(a) If C is a line then every point has N-rank 1.

(b) If C is a circle then its center has N-rank ∞ and every other point has N-rank 2.

(c) Let C be the parabola $x=t,\ y=t^2$. Then the normal line to the parabola at a point (t, t^2) is

$$y-t^2=-\frac{1}{2t}(x-t), \quad t\neq 0$$

and $x=0$ if $t=0$. Equivalently, the normal line to C at (t, t^2) for all t is

$$2t^3-2ty+t-x=0.$$

Therefore the N-rank of (x, y) is the number of distinct real roots of the reduced cubic

$$t^3+\left(\frac{1-2y}{2}\right)t-\frac{x}{2}.$$

From the theory of equations we know that a reduced cubic t^3+pt+q will have

(i) three real roots of which at least two are equal if $4p^3+27q^2=0$;
(ii) one real root if $4p^3+27q^2>0$;
(iii) three distinct real roots if $4p^3+27q^2<0$.

Applying these results to the cubic we conclude that it will have two roots or one triple root if (x, y) lies on the curve

$$4\left(\frac{1-2y}{2}\right)^3+27\left(\frac{x}{2}\right)^2=0. \tag{1}$$

The points above the curve have N-rank 3, those below have N-rank 1. The graphs of (1), the parabola, and points of various N-ranks with associated normal lines are shown in Figures 1, 2, and 3. The cusp of the curve (1) is the point $(0, 1/2)$ and at that point the reduced cubic is t^3, which has the single root $t=0$ of multiplicity 3. Consequently $(0, 1/2)$ has N-rank 1, and all other points on the curve (1) have N-rank 2.

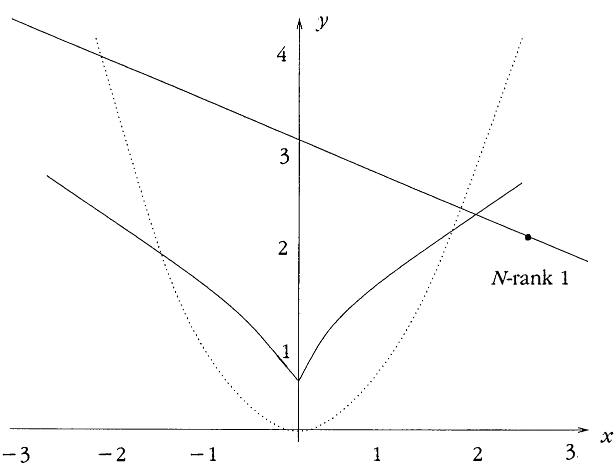

Figure 1

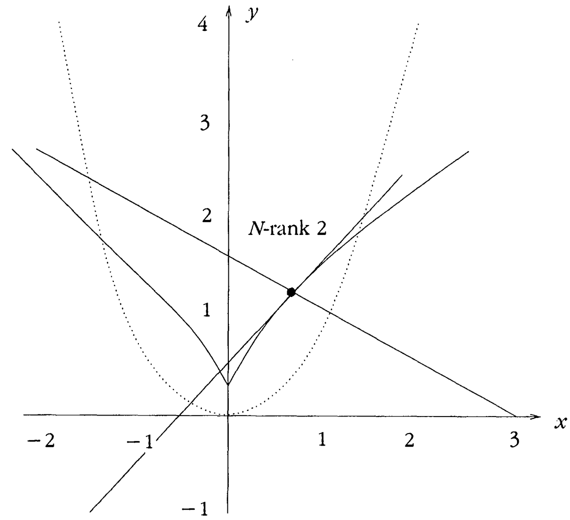

Figure 2

Assume, now, that C is the graph of a polynomial $x=t$, $y=p(t)$. Then an equation for the normal line to C at the point $(t, p(t))$ is

$$y-p(t)=-\frac{1}{p'(t)}(x-t),\quad p'(t)\neq 0$$

or $x=t$ if $p'(t)=0$. We define a function of three variables by

$$g(t,x,y)=p(t)p'(t)-p'(t)y+t-x \tag{2}$$

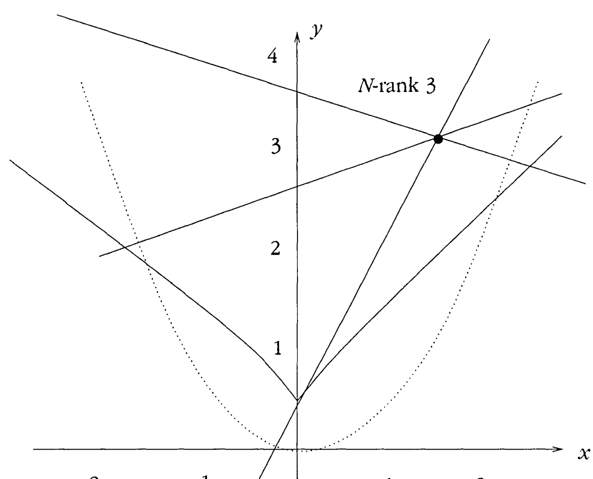

Figure 3

and note that

(i) The N-rank of (x, y) is the number of different real values of t for which $g(t, x, y) = 0$.
(ii) If the degree of $p(t)$ is n then the degree of g in t is $2n-1$.
(iii) Since g is a polynomial in t of odd degree $2n-1$, every point (x, y) has N-rank satisfying $1 \leq N\text{-rank} \leq 2n-1$.

The evolute of a curve C is defined to be the locus of the centers of curvature of C. We will show that for a polynomial, its evolute will be the boundary between the regions containing points of N-rank n for various n.

The analysis above shows that the rank of the point (x, y) is the number of different real solutions t of

$$p(t)p'(t) - p'(t)y + t - x = 0. \tag{3}$$

Fix $x = x_0$ and solve (3) for y to obtain

$$y = p(t) + \frac{t - x_0}{p'(t)}.$$

Let $s = h(t) = p(t) = \frac{t-x_0}{p'(t)}$. If x_0 does not belong to a vertical normal line to the graph of p, i.e., if $x_0 \neq t$ for all t such that $p'(t) = 0$, then the N-rank of (x_0, y) is the number of values of t such that $h(t) = y$, or equivalently the number of points of intersection of the horizontal line $s = y$ with the graph of h. If $x_0 = t$ where $p'(t) = 0$ then the N-rank of (x_0, y) is one plus the number of points of intersection of $s = y$ with the graph of h. In either case we observe that the N-rank of (x_0, y) changes only at those values of y such that the line $s = y$ is a horizontal tangent line to the graph of h at one or more relative extremal points of h. So as y varies, the N-rank of (x_0, y) changes only when $y = h(t)$ with

$$h'(t) = 0 = p'(t) + \frac{p'(t) - p''(t)(t - x_0)}{p'(t)^2}.$$

Solving for x_0 gives

$$x_0 = t - \frac{p'(t)^3}{p''(t)} - \frac{p'(t)}{p''(t)}.$$

Substitute this into (3) to obtain

$$y = p(t) + \frac{p'(t)^2}{p''(t)} + \frac{1}{p''(t)}.$$

We conclude that in moving along the vertical line $x = x_0$ the N-rank of the point (x_0, y) changes only when there exists a t such that

$$x_0 = t - \frac{p'(t)^3}{p''(t)} - \frac{p'(t)}{p''(t)},$$

$$y = p(t) + \frac{p'(t)^2}{p''(t)} + \frac{1}{p''(t)}.$$

By letting y_0 be fixed and varying x along the horizontal line $y=y_0$ we obtain the same equations as the previous ones, with x_0 replaced by x and y replaced by y_0. We therefore conclude that by varying x or y we can identify the N-rank of a point (x, y), and a change of rank can occur only when there is a t such that

$$x = t - \frac{p'(t)^3}{p''(t)} - \frac{p'(t)}{p''(t)},$$
$$y = p(t) + \frac{p'(t)^2}{p''(t)} + \frac{1}{p''(t)}. \tag{4}$$

We note that the curve (4) is the locus of points (x, y) such that $g(t, x, y)$ (see (2)) has a repeated real root. For if t is a repeated root of $g(t, x, y)$ then t is a solution to

$$g(t,x,y) = p(t)p'(t) - p'(t)y + t - x = 0,$$
$$\frac{\partial}{\partial t} g(t,x,y) = p'(t)^2 + p(t)p''(t) - p''(t)y + 1 = 0 \tag{5}$$

where necessarily $p''(t) \neq 0$, otherwise the second equation has no solution. Consequently we may solve (5) for x and y in terms of t to obtain (4).

The parametric curve (4) gives the set of points (x, y) where the polynomial $g(t, x, y)$ has repeated roots. This curve is the evolute of C (See, for example, J. Dennis Lawrence, *A Catalog of Special Plane Curves*, Dover, 1972); it is the locus of the centers of curvature of C. Our analysis shows that the evolute of the polynomial curve C: $x=t$, $y=p(t)$ gives the boundary for the regions of various N-ranks.

We now give the graphs of several polynomial curves and their evolute curves, and label the N-ranks of the various regions.

(a) Let C be the cubic $x=t$, $y=t^3$. The evolute of C is

$$x = t - \frac{27t^6 + 3t^2}{6t}, \qquad y = t^3 + \frac{1+9t^4}{6t}$$

and the graph of this curve together with C is in Figure 4. Points in region A have N-rank 3 while points in region B have N-rank 1. The two cusps occur where $\frac{dx}{dt} = 0$, i.e., where $x = \pm \frac{2}{5}\left(\frac{1}{45}\right)^{1/4}$. One identifies these as points of rank 1. All other points on the evolute curve have rank 2.

(b) Let C be the curve $x=t$, $y=t^3-3t$. The evolute of C is

$$x = t - \frac{(3t^2-3)^3 + (3t^2-3)}{6t},$$

$$y = t^3 - 3t + \frac{1+(3t^2-3)^2}{6t}$$

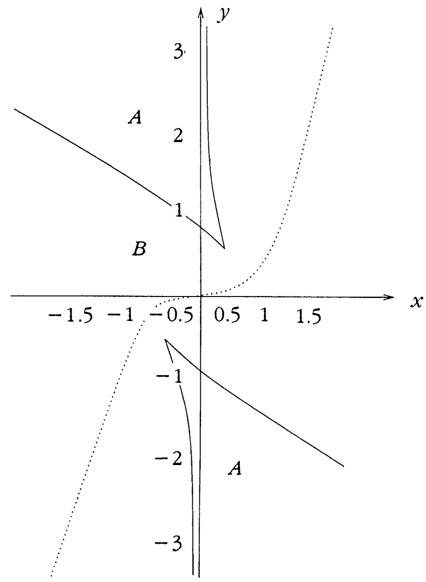

Figure 4

and its graph, along with C, is in Figure 5. By taking test points in each region we identify region A as points of rank 5, region B consists of points of rank 3 and region C contains points of rank 1. The two cusps have rank 3, as do the two points of intersection. The remaining points on the evolute curve have rank 2 (boundary points between B and C) or 4 (boundary points between A and B).

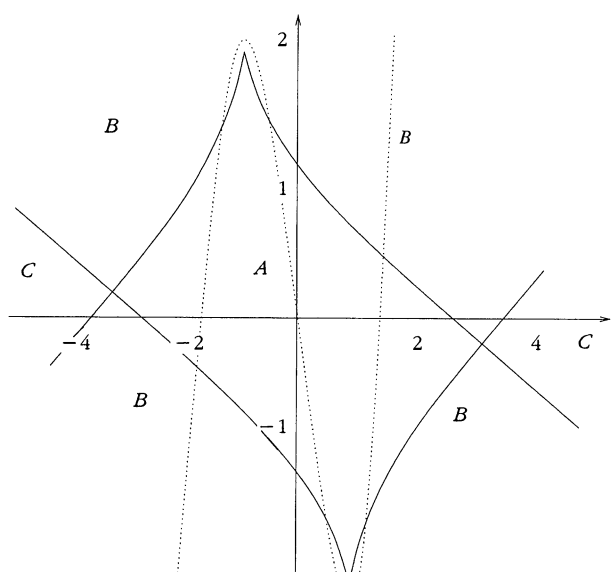

Figure 5

(c) Let C be the curve $x = t$, $y = (t^2 - 1)(t^2 - 2)$. The evolute of C is

$$x = t - \frac{(4t^3 - 6t)^3 + (4t^3 - 6t)}{12t^2 - 6},$$

$$y = t^4 - 3t^2 + 2 + \frac{1 + (4t^3 - 6t)^2}{12t^2 - 6}$$

whose graph is in Figure 6, together with C.

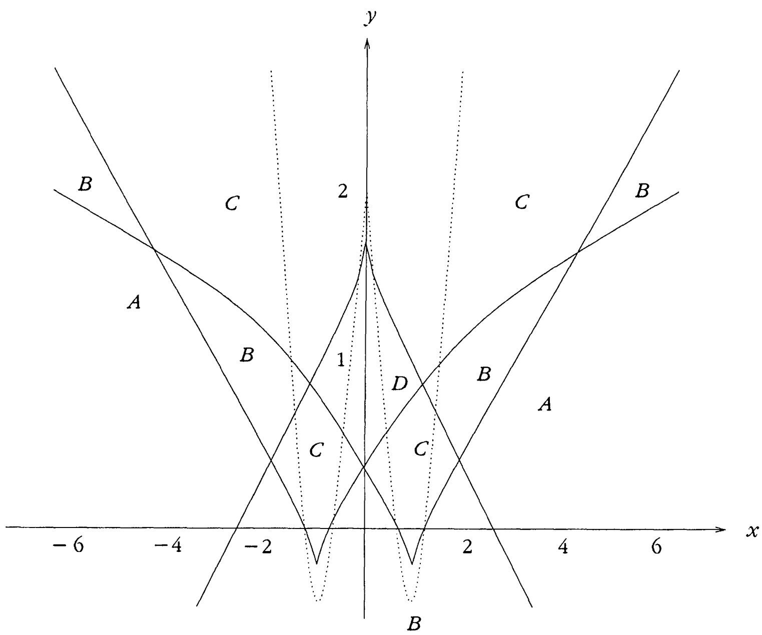

Figure 6

A: Points of rank 1
B: Points of rank 3
C: Points of rank 5
D: Points of rank 7

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A Polynomial with a Root Mod *m* for Every *m*

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If a polynomial has an integer root, of course it must have that same root mod m for every $m \in \mathbf{N}$. This issue often arises in abstract algebra where we may use the contrapositive form saying that if we can show that no solution exists mod m for some m, then there is no integer solution. For example, the equation $x^2 - 2y^2 = 3$