This phenomenon too is due to $Ax(\theta)$ sweeping out a constant area—but an area that is signed. Its sign is determined by the sign of $\det(A)$. So $Ax(\theta)$ will move in a counterclockwise direction when the two eigenvalues of A have the same sign and will move clockwise when the eigenvalues are opposite in sign.

- Try a Jordan block matrix, $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. What happens?

Note that for $A = \lambda I$, the vectors $x(\theta)$ and $Ax(\theta)$ are always collinear since any nonzero vector in $\mathbb{R}^2$ is an eigenvector. The Jordan block, however, has only one "eigen-axis", and the vectors $x(\theta)$ and $Ax(\theta)$ kiss when x is an eigenvector and then separate again, but never actually cross. After you show the graphics of this example to the students, ask them to describe what they observe and to explain the cause.

- Suppose a matrix A has complex eigenvalues. What will happen?

In this case, the vectors $x(\theta)$ and $Ax(\theta)$ never line up, but there is a point of nearest (and farthest) passage. What does the vector $x(\theta)$ represent at such a point? Does the angle between $Ax(\theta)$ and $x(\theta)$ at this point have any meaning? These are good questions for further exploration; we do not yet know their answers.

When singular values and singular vectors [**2**] have been discussed in class, their geometric implications can be illustrated with the clock hands graphics program. The lengths of the major and minor axes of the ellipse formed by $Ax(\theta)$ correspond to the maximum and minimum singular values of A, respectively. In addition, the unit vectors $x(\theta)$ at which the major and minor axes occur are the corresponding (right) singular vectors.

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Geometric Characterization of the Shortest Path in a Tetrahedron

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The problem of finding the closed path of minimum length that touches all four faces of a regular tetrahedron [**2**] was posed by Professor Igor Fedorovich Sharygin on the 1993 Moscow Mathematical Olympiad, a contest for high school students. As I was the only student to solve the problem, it remains one of my favorites. Your readers may enjoy the geometric solution below, which is a combination of Professor Sharygin's solution and mine.

Denote the vertices of the tetrahedron by ABCD, and let PQRSP be a closed path where P is on face ABC, Q is on face BCD, R is on ABD, and S is on ACD. The key lemma is this:

Lemma. *If* L *is the midpoint of* AB *and* P *is not on* CL, *there is a shorter path starting and ending on* CL *that touches all the faces.*

Proof. First note that for any four points V, W, Y, Z in space the distance between the midpoints M, N of segments VW and YZ is less than or equal to the average of the distances between the endpoints of these two segments, with equality only if the vectors joining the endpoints have the same direction:

$$\mathrm{MN} = |\mathrm{M} - \mathrm{N}| = \frac{1}{2}|\mathrm{V} + \mathrm{W} - \mathrm{Y} - \mathrm{Z}| \leq \frac{1}{2}\left(|\mathrm{V} - \mathrm{Y}| + |\mathrm{W} - \mathrm{Z}|\right) = \frac{1}{2}(\mathrm{VY} + \mathrm{WZ}).$$

Now, under the reflection in the plane CDL the path PQRSP is sent to a path $\mathrm{P'S'R'Q'P'}$ of equal length, where $\mathrm{P'}$, $\mathrm{Q'}$, $\mathrm{R'}$, and $\mathrm{S'}$ are the reflections of points P, S, R, and Q respectively. For x = P, Q, R, S, let x'' be the midpoint of x and x'. (Note that since $\mathrm{Q'}$ is the reflection of S, and $\mathrm{S'}$ is the reflection of Q, x and x' are always on the same face. Please draw a picture!) Now consider the path $\mathrm{P''Q''R''S''P''}$. By the lemma,

$$\begin{aligned}
\mathrm{P''Q''} &\leq \frac{1}{2}(\mathrm{PQ} + \mathrm{P'Q'}),\\
\mathrm{Q''R''} &\leq \frac{1}{2}(\mathrm{QR} + \mathrm{Q'R'}),\\
\mathrm{R''S''} &\leq \frac{1}{2}(\mathrm{RS} + \mathrm{R'S'}),\\
\mathrm{S''P''} &\leq \frac{1}{2}(\mathrm{SP} + \mathrm{S'P'}).
\end{aligned}$$

Summing these inequalities, we find that $\mathrm{P''Q''R''S''P''}$ is shorter than PQRSP, since the condition for equality in the lemma cannot hold for all four pairs of segments.

Applying the lemma again, we conclude that if K is the midpoint of CD, then the minimal length path PQRSP, if such a path exists, must have P on CL, Q on BK, R on DL, and S on AK. The solution now follows by observing that the shortest path is obtained by minimizing the distance between successive medians: CL to BK, BK to DL, DL to AK, and AK to CL. This is an easy exercise in analytic geometry; one finds that the minimal distance between successive medians is $1/\sqrt{10}$ times the length of a side of the tetrahedron. [*Editor's note:* Proposition 2 of [**1**] gives general formulas for the minimal distance between any two skew lines in space and also for the coordinates of the endpoints of the shortest segment joining the two lines.] The length of the shortest path in a regular tetrahedron is therefore $4/\sqrt{10}$ times the length of a side.

References

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