One proof of (*) is by induction. Clearly $(1 + x)^2 > 1 + 2x$. And if $(1 + x)^k > 1 + kx$, then

$$(1 + x)^{k+1} = (1 + x)^k(1 + x) > (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 > 1 + (k + 1)x.$$

To prove that $T_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing, consider

$$\frac{T_{n-1}}{T_n} = \left(\frac{n^2}{n^2 - 1}\right)^n \cdot \left(\frac{n}{n+1}\right) = \left(1 + \frac{1}{n^2 - 1}\right)^n \cdot \left(\frac{n}{n+1}\right)$$

for $n > 1$. By virtue of (*),

$$\left(1 + \frac{1}{n^2 - 1}\right)^n > 1 + \frac{n}{n^2 - 1} > 1 + \frac{n}{n^2} = \frac{n+1}{n},$$

we obtain

$$\frac{T_{n-1}}{T_n} > \left(\frac{n+1}{n}\right) \cdot \left(\frac{n}{n+1}\right) = 1.$$

Thus, $T_{n-1} > T_n$.

Since $S_n < T_n$ for all n, and $T_1 = 4$, it follows that S_n is bounded from above. To show that S_n is increasing, observe that

$$\frac{S_n}{S_{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n}{n-1}\right)}{\left(\frac{n}{n-1}\right)^n} = \left(\frac{n^2 - 1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right) = \left(1 - \frac{1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right)$$

for $n > 1$. Again, using (*), we get

$$\left(1 - \frac{1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right) > \left(1 - \frac{n}{n^2}\right) \cdot \left(\frac{n}{n-1}\right) = 1.$$

Hence, $S_n > S_{n-1}$.

Editor's Note: For a related discussion using the Fundamental Theorem of Calculus and the definition of e as the unique number for which $\int_1^e dx/x = 1$, see Lee Badger's classroom capsule "A Nonlogarithmic Proof that $(1 + 1/n)^n \to e$" [TYCMJ, 13(November 1982) 331–332].

Area of a Parabolic Region

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Although students can quickly recall that the area of a circle is πr^2 and the area of an ellipse is πab, there does not appear to be a standard formula that they can recall when dealing with areas of parabolic regions. Thus it may be instructive to prove the following:

The area bounded by the parabola $y = ax^2 + bx + c$ and the x-axis is

$$A = \frac{\Delta^{3/2}}{6a^2}, \tag{1}$$

when $\Delta = b^2 - 4ac$ is positive. In particular,

$$A = \frac{2}{3} BH, \tag{2}$$

where B and H, respectively, denote the base length and height of the enclosed area.

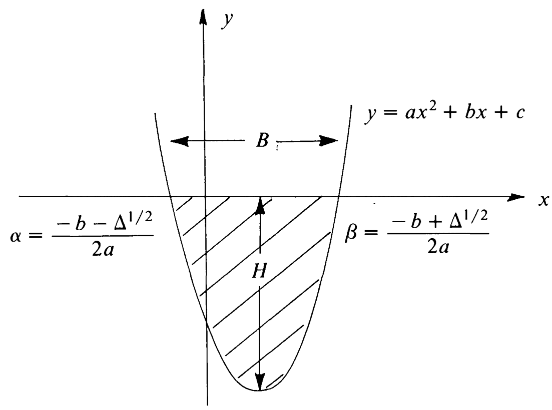

Figure 1.

The area of the parabolic region is given by:

$$A = \left|\int_\alpha^\beta (ax^2 + bx + c)\,dx\right| = \frac{a}{3}(\beta^3 - \alpha^3) + \frac{b}{2}(\beta^2 - \alpha^2) + c(\beta - \alpha). \tag{3}$$

Now from

$$\beta + \alpha = \frac{-b}{a}, \quad \beta - \alpha = \frac{\Delta^{1/2}}{a}, \quad \text{and} \quad \alpha\beta = \frac{c}{a},$$

we compute:

$$\beta^2 - \alpha^2 = \frac{-b\Delta^{1/2}}{a^2} \quad \text{and} \quad \beta^3 - \alpha^3 = (\beta - \alpha)\left[(\beta + \alpha)^2 - \alpha\beta\right] = \frac{(b^2 - ac)\Delta^{1/2}}{a^3}.$$

Substitution of these expressions into (3) yields (1). We can now easily verify (2) by completing the square

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$$

and observing that y has extremal value $\frac{-\Delta}{4a}$ when $x = \frac{-b}{2a}$. Thus, $H = \left|\frac{\Delta}{4a}\right|$. And since $B = \beta - \alpha = \frac{\Delta^{1/2}}{a}$, it is now clear that

$$A = \frac{\Delta^{3/2}}{6a^2} = \frac{2}{3}\left(\frac{\Delta^{1/2}}{a}\right)\left(\frac{\Delta}{4a}\right) = \frac{2}{3} BH.$$

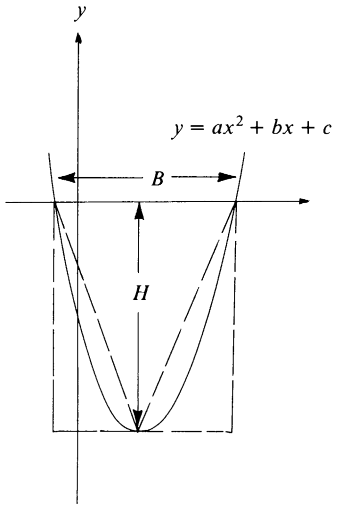

Figure 2.

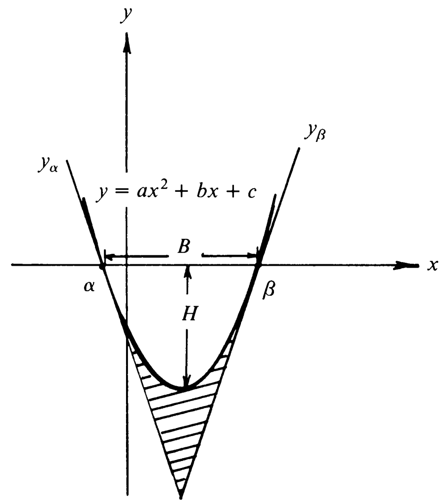

Figure 3.

Example 1. (Figure 2) Comparing the parabolic area $A_p = \frac{2}{3}BH$ with the area $A_T = \frac{1}{2}BH$ of the inscribed triangle and the area $A_R = BH$ of the circumscribed rectangle, we see that

$$A_T = \frac{3}{4}A_p, \quad A_p = \frac{2}{3}A_R, \quad \text{and} \quad A_T = \frac{1}{2}A_R.$$

Example 2. (Figure 3) Find the area between the parabola $y = ax^2 + bx + c$ and the tangents to the parabola at its roots. Since $2a\alpha = -b - \Delta^{1/2}$ and $2a\beta = -b + \Delta^{1/2}$, we see that

$$y'(\alpha) = 2a\alpha + b = -\Delta^{1/2} \quad \text{and} \quad y'(\beta) = 2a\beta + b = \Delta^{1/2}.$$

Thus, the equations of the tangent lines at $x = \alpha$ and at $x = \beta$ are, respectively,

$$y_\alpha = -\Delta^{1/2}(x - \alpha) \quad \text{and} \quad y_\beta = \Delta^{1/2}(x - \beta).$$

Since these lines intersect when $x = \dfrac{\alpha + \beta}{2}$, the triangle formed by these tangents has height $|\Delta^{1/2}(\dfrac{\alpha - \beta}{2})| = \dfrac{\Delta}{a} = 2H$. Accordingly, the shaded area is

$$A = \tfrac{1}{2}(B)(2H) - \tfrac{2}{3}BH = \tfrac{1}{3}BH$$

$$= \frac{\Delta^{3/2}}{12a^2}.$$