How Much Should You Pay for a Derivative?

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Derivatives are now traded on the stock exchanges. They are not the derivatives that we study in calculus, but rather financial derivatives: financial instruments whose values are derived from other financial instruments. A prime example is the stock option, which allows one to buy a stock on some future date at a set price (the *strike price*) no matter what the stock is actually selling for at that time. If the stock is selling for more than the strike price on the expiration date, then the option is worth the difference; otherwise it is worth nothing. What the investor must decide is, "How much should this derivative cost today?"

Those who have had an introductory probability course might say that the expected value of the option on the expiration date would be a fair price, but the answer is not so simple. Indeed, the proper pricing of options came as something of a surprise to experienced financial analysts. The theory led to the famous Black-Scholes model and formula [**1**] for which Scholes and Merton [**2, 3**] received the Nobel Prize in economics in 1997.

The analysis of the pricing of a stock option introduces students to an exciting new area of financial mathematics and also alerts them to some pitfalls in the use of expected values for decision making in practical situations. For simplicity, we will ignore such factors as transaction costs, borrowing fees, inflation, interest rates, and the possibility of selling the option before the expiration date. Here is a very simple, but illuminating, hypothetical model for the behavior of a stock price.

A fair price. Assume that the stock price is x today and on the expiration date will be either 80 or 120 dollars, each with probability $1/2$. (Perhaps the company is in competition with another company for a contract and each has probability $1/2$ of getting it.) The stock option has strike price 100. If the stock has value 120 at the expiration date, then the value of the option at that time will be $120 - 100 = 20$. If the stock has value 80 at that time, then the option will not be exercised and will have value 0. Its expected value at the expiration date is thus $20(1/2) + 0(1/2) = 10$. For someone willing to assume the risk, a price of up to \$10 would appear to be a fair amount to pay for the option. As we will see, however, this is not necessarily the case.

In deciding whether to buy an option, we must take into account what the stock is selling for, because we also have the alternative course of purchasing the stock outright. Suppose an option costs y. With that amount of dollars we could buy y/x shares of stock (fractional shares are possible). The expected value of this many shares of stock at the expiration date is $120(y/x)(1/2) + 80(y/x)(1/2) = 100y/x$. If this is more than 10, then in terms of expected values we would be better off putting the same amount of money into the stock rather than the option. That is, we should pay y for the option only if $100y/x \leq 10$ or $y \leq x/10$. From the previous analyses, a price of up to $x/10$ if $x \leq 100$, or up to 10 if $x > 100$, would appear to be a fair amount to pay for the option—but this is not the case. For $x < 100$ it turns out that $x/10$ is too much to pay and for $100 < x < 120$, 10 is too little to pay. The reason is the possibility of *arbitrage*.

Arbitrage. Arbitrage means the making of a guaranteed profit in the market by a combination of trades. As an elementary example, suppose one coin shop is selling silver dollars for \$5 each and another shop is buying them for \$6 each. A trader could buy silver dollars at the first shop and sell them to the second with a sure profit.

However, the first store would quickly begin to charge more than \$5 (since the trader would be willing to pay somewhat more) and the second store would begin offering less than \$6. Ultimately, the silver dollar prices at the two stores would converge to the same value.

Similarly, the possibility of arbitrage forces a unique price on the option, a price that does not even depend on the probabilities of the stock being worth 80 or 120 at the expiration date. We will see how arbitrage works in our example, making the unique price in fact less than $x/10$ when $80 < x < 100$ and more than 10 when $x > 100$.

To allow for arbitrage, let's consider the possibility of not only purchasing shares of stock but also borrowing money. Our borrowing will consist of selling non–interest-bearing one-dollar IOU's, payable on the expiration date of the option. We will manage to do this in such a way that the net value of stock minus debt equals the value of the option on the expiration date, no matter what the stock price is on that date.

Wheeling and dealing. Say we intend to purchase A shares of stock and sell B of the one-dollar IOU's. On the expiration date, the net worth of our holdings will be either $120A - B$ or $80A - B$, depending on whether the stock value is 120 or 80; we subtract B because we must repurchase the IOU's. To make this position equal to the option in value, we set $120A - B = 20$ and $80A - B = 0$. This gives us $A = 1/2$ and $B = 40$; in other words, we ought to buy 1/2 share of stock and sell 40 IOU's. At the end, we will still owe \$40 to buy back the IOU's, but by selling the stock we can pay for that and still have enough left over to equal the value of the option. The initial cost of this financial wheeling and dealing was $(x/2) - 40$. Recall that we received \$40 for the IOU's originally, so we subtract that from the purchase price of the stock in calculating our cost. This is balanced by carrying \$40 in debt through the period.

It follows that we should pay no more than $(x/2) - 40$ dollars for one option. Indeed, since a package of buying 1/2 share of stock and selling 40 IOU's exactly duplicates the performance of one option at the expiration date, we must pay exactly the same for an option as for the package. Otherwise, as in the coin example, a trader could go around buying options and selling packages consisting of 1/2 share of stock and \$40 of debt (or vice versa, depending on whether the option or the package is cheaper), while making a sure profit on the expiration date. Arbitrage forces the price of $(x/2) - 40$ on the option.

Moreover, it is easy to see that $(x/2) - 40 < x/10$ when $x < 100$. If $x < 80$, it would appear that one should receive money for taking the option. This is strange, but the condition $x < 80$ would never occur in reality, for one could then buy the stock with a certain profit. If $x > 100$, then the price of the option would exceed 10. This also seems paradoxical, but consider the following possible maneuver. Suppose the stock were selling for 110, and the option sold for 10. Then $(x/2) - 40 = 15$, which is 5 more than the price of the option; so one could sell 1/2 share of stock, buy 40 IOU's, buy an option, and have \$5 left over. At the expiration date, if the stock were selling for 80, one could sell the IOU's, buy back the 1/2 share of stock that had been sold, and still keep the \$5. If the stock were selling for 120, the option would be worth \$20. One could then sell the option and the 40 IOU's to buy back the 1/2 share of stock sold earlier and once again keep that \$5 profit. This shows that the arbitrage forces the price of the option to rise to \$15 in this case.

Such is the world of finance. Of course, in the real stock market, players deal with thousands of shares at a time, leading to heavy trading on option expiration dates.

Exercise. Suppose the stock is selling for \$90 a share and the option sells for \$10. Explain how a trader could make a \$5 profit with no risk.

References

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2. F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy* 81 (1973) 637–654.
3. R. C. Merton, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4 (1973) 141–183.

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Candies and Dollars

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Several years ago, my brother Ali Adnan (may God rest his soul) gave me this problem.

> John was a young boy with a jar of candies which he consumed in the following manner. On the first day, he ate one candy and gave away exactly 10% of the remaining number. On the second day, he ate two candies and gave away exactly 10% of the remaining number. He followed this pattern each day until the jar was empty. How many candies did the jar originally contain?

This problem allows for a trivial solution: the jar could have held just one candy at the start. A nontrivial solution can easily be obtained by making a guess and working backwards. Suppose we guess that, after eating candy on the next-to-last day, John gave away 10% of 10 candies; then he ate nine candies on the last day, the ninth, by definition. That is, there were 18 candies when the eighth day began, 27 candies when the seventh day began, and so on. We conclude that the jar originally contained 81 candies. This method, however, sheds no light on the surprising uniqueness of this nontrivial solution. The following argument will derive a nontrivial solution to a generalized form of this problem and also prove the uniqueness of the solution.

Let J_0 be the original number of candies (assumed to be a positive integer) and let J_r represent the number of candies in the jar at the end of the rth day. Thus, $\{J_r\}$ is to be a sequence of nonnegative integers with the property that J_r is obtained from J_{r-1} by subtracting r and then subtracting $s\%$ of what remains. In general, s does not have to be 10 (and does not have to be an integer), but we restrict it so that $n = 100/s$ is an integer. We claim that, in the unique nontrivial solution, the candies are finished off on day $n-1$ and the jar originally contained $(n-1)^2$ candies. To see this, we must analyze the relation $J_r = (J_{r-1} - r)[1 - (s/100)]$.

Setting

$$q = \left(1 - \frac{s}{100}\right)^{-1} = \left(1 - \frac{1}{n}\right)^{-1} = \frac{n}{n-1},$$

we get $J_r = (J_{r-1} - r)q^{-1}$. Thus,

$$J_{r-1} = qJ_r + r = q(qJ_{r+1} + r + 1) + r = q^2 J_{r+1} + r + (r+1)q,$$