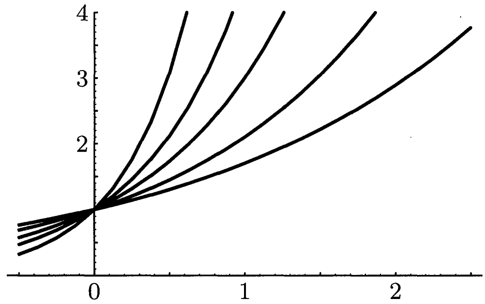

Figure 1. $y = 1.7^x$, 2.1^x, 3^x, 4.5^x, 9.5^x.

To do so, use differentiation to find the point of tangency, which, for the function $y = a^x$, is $\left(1/\ln(a), a^{1/\ln(a)}\right)$. But from properties of logarithms, it follows that $a^{1/\ln(a)} = e$ for any value of $a \neq 1$. Hence the points of tangency do indeed lie on a horizontal line, namely, the line $y = e$.

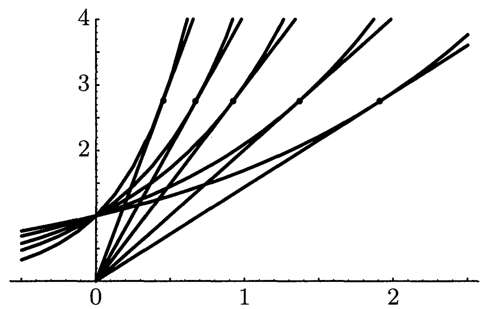

Figure 2. Points of tangency.

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A Concurrency Theorem and *Geometer's Sketchpad*

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Draw an arbitrary △ABC inside a circle, extend the sides to meet the circle, and join these points to vertices of the triangle as shown in Figure 1. The resulting lines AQ, BR, and CP always appear concurrent.

With the software *Geometer's Sketchpad* the sketch is considerably improved. All of the figures in this article were constructed by using *Sketchpad.* Not only does *Geometer's Sketchpad* allow quicker, easier, and more accurate "straightedge and compass" constructions, but it also allows these sketches to be transformed in all sorts of dynamic ways.

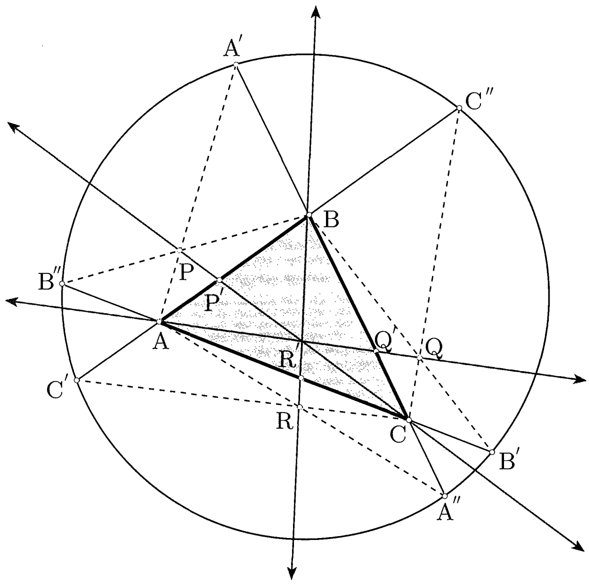

Figure 1

For instance, it is easy to change △ABC into a seemingly endless variety of shapes, sizes, and positions inside the circle. In fact, △ABC need not be restricted to the interior of the circle; see Figure 2. In each case AQ, BR, and CP appear concurrent.

These observations can be summarized as follows.

Theorem 1. *Let* △ABC *be any triangle in the plane of a circle such that* AB *intersects the circle at* C′ *and* C″, BC *intersects the circle at* A′ *and* A″, *and* CA *intersects the circle at* B′ *and* B″. *If* BB′ *and* CC″ *intersect at* Q, CC′ *and* AA″ *intersect at* R, *and* AA′ *and* BB″ *intersect at* P, *then* AQ, BR, *and* CP *are concurrent.*

We shall assume that the figures are in general position so as to avoid parallel lines and other special cases. In order to prove Theorem 1 we need a famous result

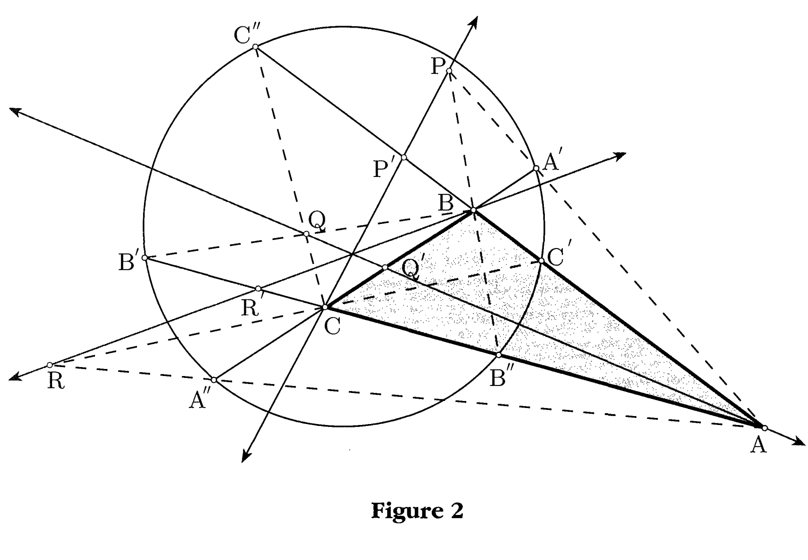

Figure 2

published in 1678 by the Italian mathematician Giovanni Ceva (c. 1647–1736). Figure 3 shows two illustrations of Ceva's theorem with the point of concurrency inside or outside the triangle. The proof of this theorem can be found in *A Survey of Geometry* by Howard Eves (Allyn and Bacon, Boston, rev. ed., 1972, pp. 63–66) and in many college geometry textbooks.

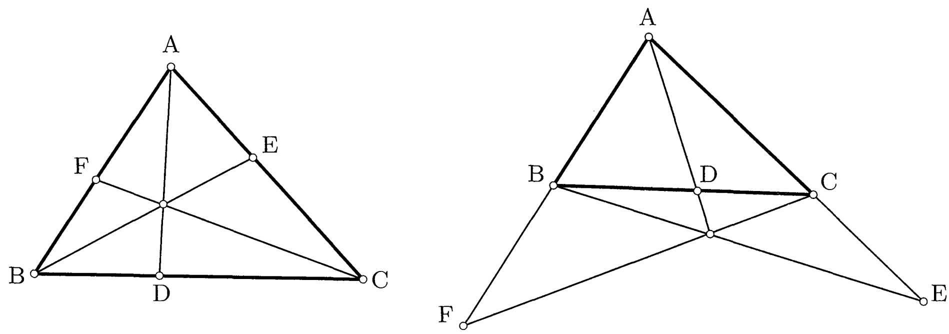

Figure 3. Two illustrations of Ceva's theorem.

Theorem 2. (Ceva's theorem.) *If* F, E, *and* D *are points (except vertices) of the (possibly extended) sides* AB, BC, *and* CA *of* $\triangle$ABC, *respectively, such that* AD, BE, *and* CF *are concurrent, then* $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$, *and conversely.*

Returning to the proof of Theorem 1, we let Q′ be the intersection of AQ and BC, R′ the intersection of BR and CA, and P′ the intersection of CP and AB in Figures 1 and 2. By using Ceva's theorem three times on $\triangle$ABC with points of concurrency Q, R, and P, we have

$$\frac{AB'}{B'C} \cdot \frac{CQ'}{Q'B} \cdot \frac{BC''}{C''A} = 1, \quad \frac{BC'}{C'A} \cdot \frac{AR'}{R'C} \cdot \frac{CA''}{A''B} = 1, \quad \text{and} \quad \frac{CA'}{A'B} \cdot \frac{BP'}{P'A} \cdot \frac{AB''}{B''C} = 1,$$

respectively. Multiplying these three equations together and rearranging factors gives

$$\left(\frac{AR'}{R'C} \cdot \frac{CQ'}{Q'B} \cdot \frac{BP'}{P'A}\right)\left(\frac{AB'}{AC'} \cdot \frac{AB''}{AC''}\right)\left(\frac{BC'}{BA'} \cdot \frac{BC''}{BA''}\right)\left(\frac{CA'}{CB'} \cdot \frac{CA''}{CB''}\right) = 1. \qquad (1)$$

Next we consider vertex A. Since the products of segments of two chords intersecting in a circle are equal (see Figure 1), or since the products of secants and their external segments of two secants intersecting outside a circle are equal (Figure 2), then $AB' \cdot AB'' = AC' \cdot AC''$. Hence $\frac{AB'}{AC'} \cdot \frac{AB''}{AC''} = 1$. Similarly, $\frac{BC'}{BA'} \cdot \frac{BC''}{BA''} = 1$ and $\frac{CA'}{CB'} \cdot \frac{CA''}{CB''} = 1$. Substitution into equation (1) and simplifying yields $\frac{AR'}{R'C} \cdot \frac{CQ'}{Q'B} \cdot \frac{BP'}{P'A} = 1$. By the converse portion of Ceva's theorem we have that AQ, BR, and CP are concurrent. This completes the proof of Theorem 1. □

Figure 4 is essentially the same as Figure 1 except that points B′ and B″ are renamed as B″ and B′, respectively, which yields a different point of concurrency. We can also interchange A′ and A″, and C′ and C″ in various ways, so that there are actually eight points of concurrency for a given circle and triangle.

Can Theorem 1 be generalized to other polygons in the plane of a circle?

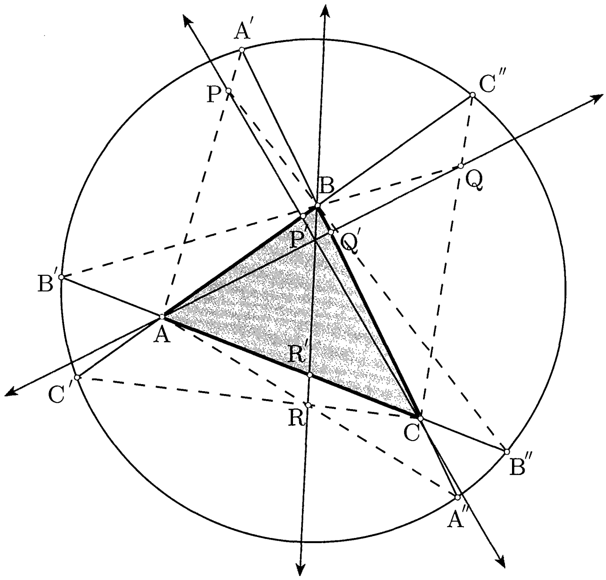

Figure 4

Figure 5 shows that it is possible to have concurrency for an irregular pentagram constructed inside a circle like the triangle in Figure 1. Is this merely a coincidence, or is a more general pattern lurking about?

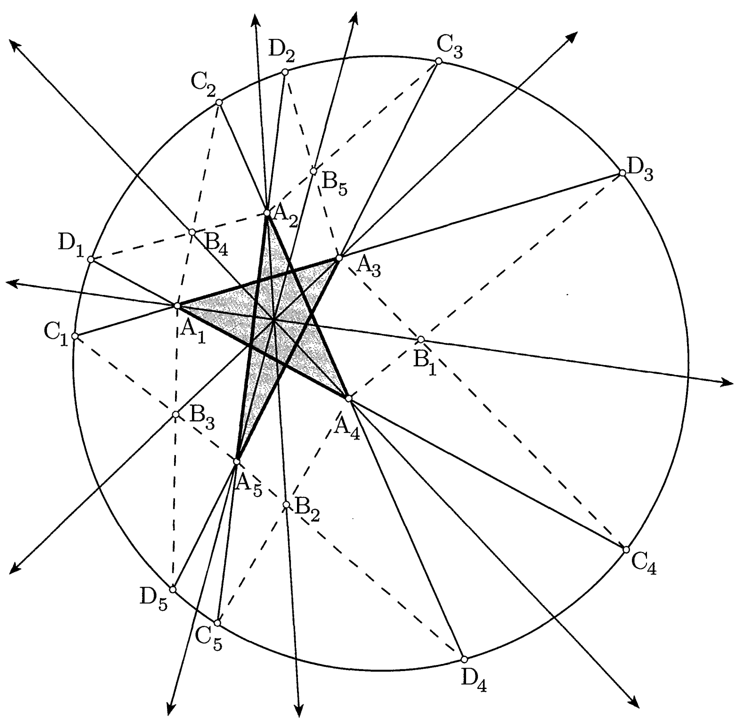

Figure 5