Tying Up Loose Ends with Probability

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All of us seek activities that will capture our students' interest. The following in-class activity, which can be used in such courses as finite mathematics, introductory statistics, or mathematics for elementary teachers, does just that. When I first saw it presented at the National Council of Teachers of Mathematics "Connections Seminar" (Tampa, FL, 1995), no analysis of the experimental outcomes was done. My adaptation, which I call *Loopers*, includes such an analysis, and it has been a great hit with my probability students, most of whom are majoring in mathematics, computer science, or engineering and have at least nine credits of calculus. One actually exclaimed, "This is the neatest thing I have ever done in a math class!" Here is how *Loopers* works.

How many loops? First, I divide the class into pairs, giving each pair six pieces of string about 12 inches long. One partner drapes all six strings over the palm of one hand, making a fist to secure them (see Figure 1, left). The second partner takes the ends of the strings two by two and ties them together, thus producing three knots on one side of the fist and three knots on the other side (Figure 1, right). While all fists are still firmly clenched, I ask: "What shapes might you expect to see when you hold out your strings?"

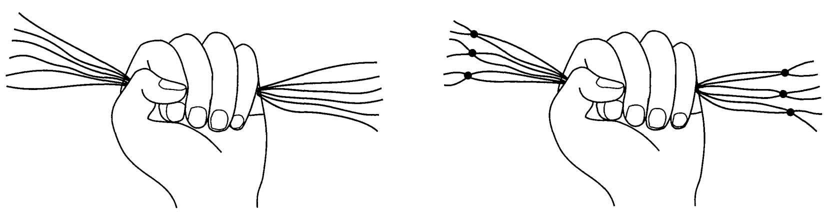

Figure 1

After talking it over, they draw their conclusions on the board and eventually discover (with hints, if needed) that only three possibilities exist: one closed ring, two rings, and three rings. (Note that we are not distinguishing between interlocked rings and separated rings; the primary focus is the *number of loops*). Now I have them predict which possibility is most likely to occur and which is least likely. After ample time for discussion, I let them open their fists and show the results. We then record the experimental probabilities of forming one closed ring, two rings, and three rings.

Students of probability should understand the difference between *experimental* relative frequency and *theoretical* probability. *Loopers* demonstrates this difference, since in a class of about 30 students the relative frequencies obtained (the ratio of the number of observed occurrences of the event to the total number of trials) can be quite different from the theoretical probabilities of forming one, two, and three rings (respectively, $8/15$, $6/15$, and $1/15$). I point out that if our class were larger or we repeated the experiment several times, the relative frequencies would get closer and closer to the theoretical probabilities.

Exercise. *How were these theoretical probabilities obtained? Give a written explanation.*

I give my students about a week to work on figuring this out. Here is one of several possible solutions.

Solution. Since the outcome depends only on how the last three knots are tied, consider the first three knots fixed. Arrange the strings side by side with the knotted ends on top, and label the six free ends as in Figure 2.

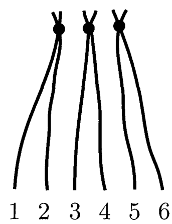

Figure 2

First, we count the total number of outcomes. Choose any end; five ends are left to which it could be tied. Tie it to one of them, and choose another end. Now only three ends remain to which this end could be tied. Tie it to one of them, and only one choice remains for forming the last pair. By the basic principle of counting, the total number of outcomes is $5 \times 3 \times 1 = 15$.

Now, there is only one way to get three loops: by tying end 1 to end 2, end 3 to 4, and end 5 to 6. So the probability of forming three rings is $1/15$.

How many ways are there to form one loop? Choose an end, say end 1. We can't tie it to end 2, or we would get more than one loop. That leaves four ends to pick from, so say we tie end 1 to end 3. Now pick another end, say 2. End 2 cannot be tied to end 4 or again we would get more than one loop; so we have only two remaining ends to which we can tie end 2. Finally, there is only one possibility for the last pair. Therefore the total number of ways to form one ring is $4 \times 2 \times 1 = 8$. Thus, the probability of forming one ring is $8/15$.

Last, to form two loops we must tie both ends of one string together. There are three ways to do this (1 tied to 2, 3 to 4, or 5 to 6). We pick one of these choices and then consider the lowest numbered end remaining. This end must be tied to one of two ends (it can't be tied to the other end of the string), and then again the last pair is fixed. So the number of ways to form two rings is $3 \times 2 \times 1 = 6$, which shows that the probability of forming two rings is $6/15$. $\square$

Loopers has many benefits. The in-class experiment itself captures my students' interest because they are immediately curious about which outcome is most likely to occur and why. Because the written assignment of calculating the theoretical probabilities can be approached in several ways, the students can later compare their methods of solution and reason about their different strategies. Moreover, requiring a written explanation lets me assess whether they know how to arrive at the correct answer as well as how clearly they can explain it. Once their solutions are graded and returned, I allow class time for sharing results, so the students can see methods other than their own. While some students simply list all possible outcomes, others find methods similar to the one above to calculate the probabilities.

Loopers could be adapted to any grade level from elementary school through college. Even young children (provided they know how to tie knots!) can perform a simplified four-string experiment and follow a discussion about which outcome is more (or less) likely [see L. S. Sgroi and R. J. Sgroi, *Mathematics for Elementary School Teachers: Problem-solving Investigations*, PWS-Kent, Boston, 1993]. In higher grades, a more formal analysis like mine may be appropriate, perhaps even generalizing to a greater number of strings.

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Unifying a Family of Extrema Problems

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One of our goals as mathematics teachers is to help students see how to bring unity to problems that seem at first glance to be rather different. In this article we show how Lagrange multipliers can be used to solve a general problem that unifies several common Calculus I and Calculus III extrema problems. The solutions to these extrema problems share an interesting feature which we think is not widely recognized. In addition, we exhibit an elegant proof of the arithmetic–geometric means inequality.

Fence problem. We begin with a Calculus I problem that illustrates some of the features of the more general problem. We seek to minimize the total cost of fencing to enclose a rectangular area of 5184 square meters. The cost of fencing varies by sides as follows: \$4.00 per meter and \$5.00 per meter along the two north-south boundaries and \$1.50 per meter and \$2.50 per meter along the two east-west boundaries.

This problem is easily solved by standard methods; we will not repeat the solution here. However, a graphical analysis is instructive.

Let x be the length in the north-south direction and $C(x)$ be the total cost. Also, let $S(x)$ be the cost of the north-south fencing, and $W(x)$ be the cost of the east-west fencing. Then $S(x) = 9x$, $W(x) = 4(5184/x)$, and $C(x) = S(x) + W(x)$. As Figure 1 illustrates, $C(x)$ is a minimum when $S(x) = W(x)$. In other words, the

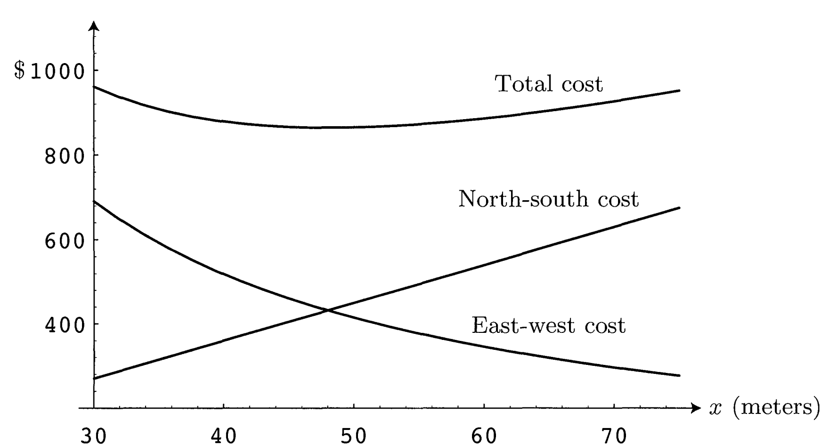

Figure 1