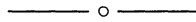


geometric proofs like that above appear in Bary [2]. Bary credits the geometric approach to Denjoy. The theorem appears in several standard textbooks, such as Rudin's *Principles of Mathematical Analysis* and Bartle's *Elements of Real Analysis*.

References

1. N. H. Abel, Note sur le mémoire de Mr. L. Oliver No. 4, du second tome de ce journal, ayant pour titre "remarques sur les séries infinies et leur convergence," *Crelles Journal* 3 (1828) 80. See also: Sur les séries, *Oeuvres complètes*, vol. 2, 2nd ed., Christiania, CITY??, 1881, pp. 197–201.
2. N. Bary, *A Treatise on Trigonometric Series*, Pergamon, London, 1961, pp. 470–471.
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A Note on Taylor's Series for $\sin(ax + b)$ and $\cos(ax + b)$

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For many students in elementary calculus, finding the Taylor series expansion of $\sin(ax + b)$ and $\cos(ax + b)$ (where a and b are constants with $a \neq 0$) centered at some point $x_0 \neq 0$ is quite formidable. Most calculus textbooks contain few exercises to find such expansions, and these exercises typically ask students to find only the first few terms of these expansions. The purpose of this note is to develop a technique that will allow students to easily determine the complete Taylor expansions.

In elementary calculus, it is shown that

$$\frac{d^n \sin x}{dx^n} = \begin{cases} \sin x, & \text{if } n \equiv 0 \pmod{4} \\ \cos x, & \text{if } n \equiv 1 \pmod{4} \\ -\sin x, & \text{if } n \equiv 2 \pmod{4} \\ -\cos x, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Using the addition formulas for the sine and cosine, it is straightforward to show that this can be written simply as

$$\frac{d^n \sin x}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right). \quad (1)$$

Similarly,

$$\frac{d^n \cos x}{dx^n} = \cos\left(x + \frac{n\pi}{2}\right). \quad (2)$$

So, by the chain rule and identities (1) and (2),

$$\frac{d^n \sin(ax + b)}{dx^n} = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

and

$$\frac{d^n \cos(ax + b)}{dx^n} = a^n \cos\left(ax + b + \frac{n\pi}{2}\right).$$

As a result, the Taylor series expansions centered at x_0 for $\sin(ax+b)$ and $\cos(ax+b)$ are

$$\sin(ax+b) = \sum_{n=0}^{\infty} \frac{a^n \sin\left(ax_0 + b + \frac{n\pi}{2}\right)}{n!} (x-x_0)^n \quad (3)$$

and

$$\cos(ax+b) = \sum_{n=0}^{\infty} \frac{a^n \cos\left(ax_0 + b + \frac{n\pi}{2}\right)}{n!} (x-x_0)^n \quad (4)$$

for $x \in (-\infty, \infty)$.

Example 1. From (3), the Taylor series centered at $x_0 = \pi/6$ for $\sin x$ is

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{\sin\left(\frac{\pi}{6} + \frac{n\pi}{2}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(\cos \frac{n\pi}{2} + \sqrt{3} \sin \frac{n\pi}{2}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n. \end{aligned}$$

Example 2. From (4), the Taylor series centered at $x_0 = \pi/6$ for $\cos(3x + \pi/4)$ is

$$\begin{aligned} \cos\left(3x + \frac{\pi}{4}\right) &= \sum_{n=0}^{\infty} \frac{3^n \cos\left(\frac{3\pi}{6} + \frac{\pi}{4} + \frac{n\pi}{2}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n \\ &= \frac{-1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{3^n}{n!} \left(\cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right) \left(x - \frac{\pi}{6}\right)^n. \end{aligned}$$

Finally, it should be noted that the Taylor series expansion centered at x_0 for $\sin(ax+b)$ can be obtained by rewriting $\sin(ax+b)$ and using the well-known Maclaurin expansions of $\sin x$ and $\cos x$ as follows.

$$\begin{aligned} \sin(ax+b) &= \sin[a(x-x_0) + (ax_0+b)] \\ &= \sin[a(x-x_0)] \cos(ax_0+b) + \cos[a(x-x_0)] \sin(ax_0+b) \\ &= \cos(ax_0+b) \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} (x-x_0)^{2n+1} \\ &\quad + \sin(ax_0+b) \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} (x-x_0)^{2n}. \end{aligned} \quad (5)$$

The interested reader may wish to show that (3) and (5) are equivalent. However, the expansion in (3) seems to be more natural and elegant than (5). Of course, a formula similar to (5) can be obtained for $\cos(ax+b)$.

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