geometric proofs like that above appear in Bary [**2**]. Bary credits the geometric approach to Denjoy. The theorem appears in several standard textbooks, such as Rudin's *Principles of Mathematical Analysis* and Bartle's *Elements of Real Analysis*.

References

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2. N. Bary, *A Treatise on Trigonometric Series*, Pergamon, London, 1961, pp. 470–471.
3. P. du Bois Reymond, Neue theorie der convergenz und divergenz von reihen mit positiven gliedern, *Crelles Journal* 76 (1873) 61–91.

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A Note on Taylor's Series for sin (*ax* + *b*) and cos (*ax* + *b*)

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For many students in elementary calculus, finding the Taylor series expansion of sin $(ax + b)$ and cos $(ax + b)$ (where a and b are constants with $a \neq 0$) centered at some point $x_0 \neq 0$ is quite formidable. Most calculus textbooks contain few exercises to find such expansions, and these exercises typically ask students to find only the first few terms of these expansions. The purpose of this note is to develop a technique that will allow students to easily determine the complete Taylor expansions.

In elementary calculus, it is shown that

$$\frac{d^n \sin x}{dx^n} = \begin{cases} \sin x, & \text{if } n \equiv 0 \pmod 4 \\ \cos x, & \text{if } n \equiv 1 \pmod 4 \\ -\sin x, & \text{if } n \equiv 2 \pmod 4 \\ -\cos x, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Using the addition formulas for the sine and cosine, it is straightforward to show that this can be written simply as

$$\frac{d^n \sin x}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right). \tag{1}$$

Similarly,

$$\frac{d^n \cos x}{dx^n} = \cos\left(x + \frac{n\pi}{2}\right). \tag{2}$$

So, by the chain rule and identities (1) and (2),

$$\frac{d^n \sin(ax+b)}{dx^n} = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

and

$$\frac{d^n \cos(ax+b)}{dx^n} = a^n \cos\left(ax + b + \frac{n\pi}{2}\right).$$

As a result, the Taylor series expansions centered at x_0 for $\sin(ax+b)$ and $\cos(ax+b)$ are

$$\sin(ax+b) = \sum_{n=0}^{\infty} \frac{a^n \sin\left(ax_0 + b + \frac{n\pi}{2}\right)}{n!} (x - x_0)^n \tag{3}$$

and

$$\cos(ax+b) = \sum_{n=0}^{\infty} \frac{a^n \cos\left(ax_0 + b + \frac{n\pi}{2}\right)}{n!} (x - x_0)^n \tag{4}$$

for $x \in (-\infty, \infty)$.

Example 1. From (3), the Taylor series centered at $x_0 = \pi/6$ for $\sin x$ is

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{\sin\left(\frac{\pi}{6} + \frac{n\pi}{2}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(\cos\frac{n\pi}{2} + \sqrt{3}\sin\frac{n\pi}{2}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n.\end{aligned}$$

Example 2. From (4), the Taylor series centered at $x_0 = \pi/6$ for $\cos(3x + \pi/4)$ is

$$\begin{aligned}\cos\left(3x + \frac{\pi}{4}\right) &= \sum_{n=0}^{\infty} \frac{3^n \cos\left(\frac{3\pi}{6} + \frac{\pi}{4} + \frac{n\pi}{2}\right)}{n!} \left(x - \frac{\pi}{6}\right)^n \\ &= \frac{-1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{3^n}{n!} \left(\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) \left(x - \frac{\pi}{6}\right)^n.\end{aligned}$$

Finally, it should be noted that the Taylor series expansion centered at x_0 for $\sin(ax+b)$ can be obtained by rewriting $\sin(ax+b)$ and using the well-known Maclaurin expansions of $\sin x$ and $\cos x$ as follows.

$$\begin{aligned}\sin(ax+b) &= \sin\left[a(x-x_0) + (ax_0+b)\right] \\ &= \sin\left[a(x-x_0)\right]\cos(ax_0+b) + \cos\left[a(x-x_0)\right]\sin(ax_0+b) \\ &= \cos(ax_0+b) \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} (x-x_0)^{2n+1} \\ &\quad + \sin(ax_0+b) \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} (x-x_0)^{2n}.\end{aligned} \tag{5}$$

The interested reader may wish to show that (3) and (5) are equivalent. However, the expansion in (3) seems to be more natural and elegant than (5). Of course, a formula similar to (5) can be obtained for $\cos(ax+b)$.

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