Where There is Pattern, There is Significance

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W. W. Sawyer [*Prelude to Mathematics*, Penguin, 1959, pp. 60–61] offers without proof a pretty pattern for trigonometry students to ponder.

His pattern is based on the identity for $\tan(A+B)$ and he made it easy to see by setting $\tan A = t$, then

$$\tan 2A = \frac{2t}{1-t^2}$$

$$\tan 3A = \frac{3t - t^3}{1 - 3t^2}$$

$$\tan 4A = \frac{4t - 4t^3}{1 - 6t^2 + t^4}.$$

It would seem that if $\tan A = t$, $\tan(nA)$ bears a remarkable relation to $(1+t)^n$. Sawyer said that a search for an underlying reason leads to Euler's formula for $e^{i\theta}$. A student who has expanded $(1+it)^4$ and rewritten the result in the form $f(t) + ig(t)$ might guess that

$$\tan(4A) = \frac{\operatorname{Im}(1+it)^4}{\operatorname{Re}(1+it)^4}.$$

A student who has seen DeMoivre's theorem has a chance to discover the argument:

$$\begin{aligned}(1+it)^n = (1 + i\tan A)^n &= \left(\frac{\cos A + i \sin A}{\cos A}\right)^n \\ &= \frac{\cos(nA)}{\cos^n A} + i\frac{\sin(nA)}{\cos^n A},\end{aligned}$$

hence

$$\frac{\operatorname{Im}(1+it)^n}{\operatorname{Re}(1+it)^n} = \frac{\sin(nA)}{\cos(nA)} = \tan(nA).$$

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Discovery or Invention?

He holds, as so many great mathematicians have, the Platonist point of view. For him mathematical truths are descriptions of a suprasensible reality, an objective reality that exists independent of our perceptions of it. The moons of Jupiter were circling in their orbits before Galileo put the telescope to his eye,...and mathematical truths are there for the mathematicians to see. (How see? Through what faculty? Spinoza said the eyes of the mind are proofs, but [he] regards proofs more in the way of spectacles, bringing the visions of intuition into sharper focus.)

Rebecca Goldstein, *The Mind-Body Problem: A Novel*, Dell Books