Problems 1–3 follow from theorems proved in [**1**], using the inequality of the arithmetic and geometric means. Simmons [**2**] includes a proof of the arithmetic–geometric means inequality as an exercise in his section on Lagrange multipliers.

References

1. Ivan Niven, *Maxima and Minima Without Calculus* (Dolciani Mathematical Expositions), Mathematical Association of America, Washington, DC, 1981.
2. George F. Simmons, *Calculus with Analytic Geometry*, 2nd ed., McGraw-Hill, New York, 1996.

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A Note on the Ratio of Arc Length to Chordal Length

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Suppose f is a differentiable function on $[a, b]$, and consider two points P and Q on the graph of $y = f(x)$. Let $L(\mathrm{P}, \mathrm{Q})$ denote the arc length of the portion of the graph between P and Q, and let $D(\mathrm{P}, \mathrm{Q})$ denote the Euclidean distance between P and Q. *Calculus*, the popular textbook by D. Hughes-Hallett et al. [Wiley, New York, 1994], gives an informal justification that the derivative of the sine function is the cosine function, based on the assumption that the ratio $L(\mathrm{P}, \mathrm{Q})/D(\mathrm{P}, \mathrm{Q})$ tends to unity as Q tends toward P along a circular arc.

Is this assumption generally true? As we shall see, it holds if f has a continuous derivative. However, the ratio need not tend to unity without this or some other restriction on f. I applaud the success of these authors in exposing the ideas of calculus—a success due in part to their willingness to forego unhelpful rigor. But I think instructors should be aware of this gap in the exposition.

Theorem. *If f has a continuous derivative on $[a, b]$, and if* P *and* Q *are points on the graph of $y = f(x)$, then*

$$\lim_{\mathrm{Q}\to\mathrm{P}} \frac{L(\mathrm{P}, \mathrm{Q})}{D(\mathrm{P}, \mathrm{Q})} = 1.$$

Proof. Without loss of generality, we may assume that $[a, b] = [0, 1]$, $f(0) = 0$, and P is the origin. We then have

$$\begin{aligned}
\lim_{\mathrm{Q}\to\mathrm{P}} \frac{L(\mathrm{P}, \mathrm{Q})}{D(\mathrm{P}, \mathrm{Q})} &= \lim_{x\to 0} \frac{\int_0^x \sqrt{1 + y'^2}\, dt}{\sqrt{x^2 + y^2}} \\
&= \lim_{x\to 0} \frac{\sqrt{1 + y'^2}}{\frac{1}{2}(x^2 + y^2)^{-1/2}(2x + 2yy')} && \text{(l'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{\sqrt{(1 + y'^2)\,(x^2 + y^2)}}{x + yy'}
\end{aligned}$$

$$= \lim_{x\to 0} \frac{\sqrt{(1+y'^2)\left(1+\left(\frac{y}{x}\right)^2\right)}}{1+\left(\frac{y}{x}\right)y'}$$

$$= \frac{1+f'(0)^2}{1+f'(0)^2} \qquad \text{(by continuity of } f'\text{)}$$

$$= 1. \qquad \square$$

Here is an example showing that the assumption of a *continuous* derivative cannot be dropped from the theorem. Define f on $[0,1]$ by the rule

$$f(x) = \begin{cases} x^2 \sin(2\pi/x), & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly, f is differentiable on $[0,1]$, with $f'(0^+) = 0$. For $x \neq 0$, a direct calculation of $f'(x)$ gives

$$f'(x) = 2x \sin\frac{2\pi}{x} - 2\pi \cos\frac{2\pi}{x}.$$

As x approaches zero in this expression, the first term tends to zero and the second term oscillates between 2π and -2π, so f' is not continuous at $x = 0$. However, f' is continuous on $(0,1]$, which implies that the arc length integral exists.

Let P denote the origin, and let $Q_m = (1/m, 0)$. I will show that $L(P, Q_m) > 3D(P, Q_m)$, for all natural numbers m. Then the conclusion of the theorem cannot hold, since the points Q_m lie on the graph of f, $Q_m \to P$ as $m \to \infty$, and $L(P, Q_m)/D(P, Q_m) > 3$.

Note that the successive positive zeros of f occur where $2\pi/x = n\pi$; that is, at the points $x_n = 2/n$. Thus f has constant sign on each interval (x_{n+1}, x_n). If $y_n = 2/(n+\frac{1}{2})$, so $x_{n+1} < y_n < x_n$ and $\sin(2\pi/y_n) = (-1)^n$, then it is clear from Figure 1 that the arc length over (x_{n+1}, x_n) is bounded below by $2|f(y_n)|$.

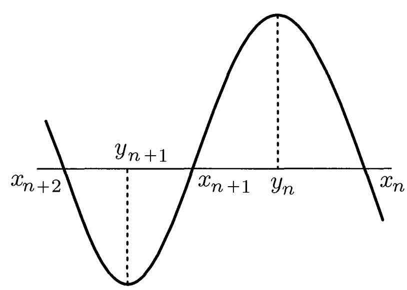

Figure 1

Then

$$L(x_{n+1}, x_n) \geq 2|f(y_n)| = 2\left(\frac{2}{n+\frac{1}{2}}\right)^2 = \frac{4n(n+1)}{\left(n+\frac{1}{2}\right)^2}\left(\frac{2}{n(n+1)}\right)$$

$$= \frac{4n(n+1)}{\left(n+\frac{1}{2}\right)^2}(x_n - x_{n+1}).$$

Now it is easy to verify that

$$\frac{4n(n+1)}{\left(n+\frac{1}{2}\right)^2} > 3$$

for all $n \geq 2$. Therefore, summing over all the intervals between successive zeros of f, from P to $\mathrm{Q}_m = x_{2m}$, we get

$$L(\mathrm{P}, \mathrm{Q}_m) = \sum_{n \geq 2m} L(x_{n+1}, x_n) > \sum_{n \geq 2m} 3(x_n - x_{n+1}) = 3x_{2m} = 3D(\mathrm{P}, \mathrm{Q}_m),$$

as claimed.

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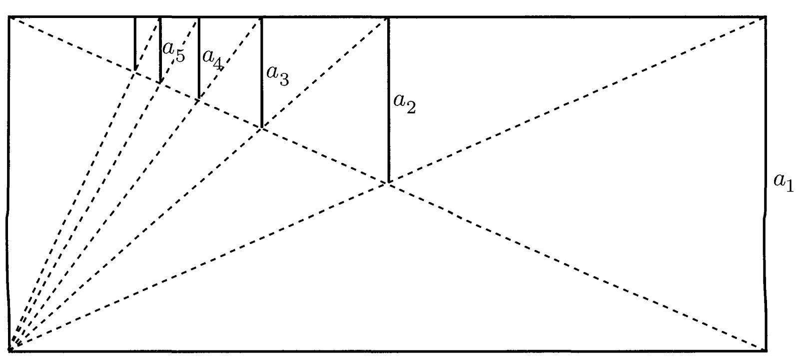
Can You Sum This Familiar Series?



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