CLASSROOM CAPSULES

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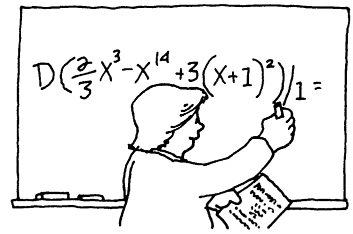

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia or Tom Farmer.

Symmetry and Integration

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Problem A-3 on the 1980 Putnam exam asked contestants to evaluate

$$\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}.$$

It appears that most of the contestants found this to be a rather difficult problem, for of the top 207 (of 2043) contestants, 141 did not submit a solution, and only 23 scored 3 or more points (out of a possible 10) [L. F. Klosinski, G. L. Alexanderson, and A. P. Hillman, The William Lowell Putnam Mathematical Competition, *American Mathematical Monthly* 88 (1981), 605–612].

But I think if this problem were given to today's students, armed with graphing calculators, the performance would be quite different. Upon seeing that the graph of the function in the integrand looks something like Figure 1, many students could now exploit the symmetry and proceed to show analytically that the value of the integral must be $\pi/4$.

Of course, using symmetry to simplify integration problems is nothing new. Virtually every calculus text published in the past 20 years has, when considering

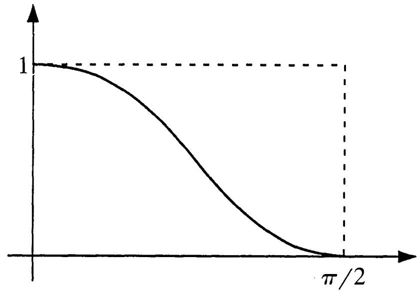

Figure 1
The graph of $y = 1/[1+(\tan x)^{\sqrt{2}}]$ on $[0, \pi/2]$.

integrals of even and odd functions, used the symmetry they exhibit to simplify the integration—e.g., if f is odd and continuous on an interval (of finite length) centered at 0, then the integral of f over that interval is 0. But functions exhibiting the sort of symmetry in the above problem are rarely considered, even in the new generation of calculus texts which make extensive use of graphing technology. With a graphing calculator, it is easy for the student to discover symmetry such as that exhibited in the Putnam problem cited above. My purpose here is to expand upon the usual notions of symmetry encountered in calculus in order to deal with such problems.

The symmetry of the function in Figure 1 is essentially the symmetry of an odd function, but with the "center" of symmetry at a point other than the origin. To be precise, we say that the graph of a plane curve is *symmetric with respect to the point* (c, d) iff whenever $(c-x, d-y)$ is on the curve, so is $(c+x, d+y)$. When the curve is the graph of a function $y=f(x)$, this condition can be expressed as $f(c-x)+f(c+x)=2f(c)$ (when $c, c-x$, and $c+x$ are all in the domain of the function), as illustrated in Figure 2.

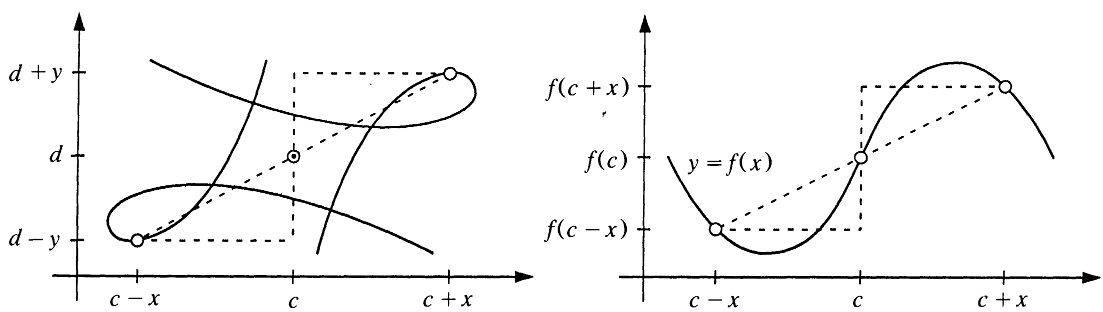

Figure 2
Symmetry with respect to a point.

Now suppose that f is continuous on the interval $[a, b]$ and that the graph of f is symmetric with respect to the point whose x-coordinate is the midpoint $(a+b)/2$ of $[a, b]$; that is, f satisfies $f(x)+f(a+b-x)=2f((a+b)/2)$ for all x in $[a, b]$. Such functions are easy to integrate:

Theorem. *Suppose f is continuous on $[a, b]$ and that $f(x)+f(a+b-x)$ is constant for all x in $[a, b]$. Then*

$$\int_a^b f(x)\,dx = (b-a)f\left(\frac{a+b}{2}\right) = \tfrac{1}{2}(b-a)[f(a)+f(b)].$$

An analytic proof is straightforward—simply split the interval of integration at its midpoint and use the substitution $u=a+b-x$ in the second integral. But perhaps more instructive is the "proof without words" in Figure 3 (page 42).

As a simple example, consider $\int_0^{\pi/2} \sin^2 x\,dx$. With a graphing calculator, the student readily notices symmetry with respect to $(\pi/4, 1/2)$, and this is easily verified: $\sin^2 x + \sin^2[(\pi/2)-x] = 1$. Thus $\int_0^{\pi/2} \sin^2 x\,dx = \pi/4$.

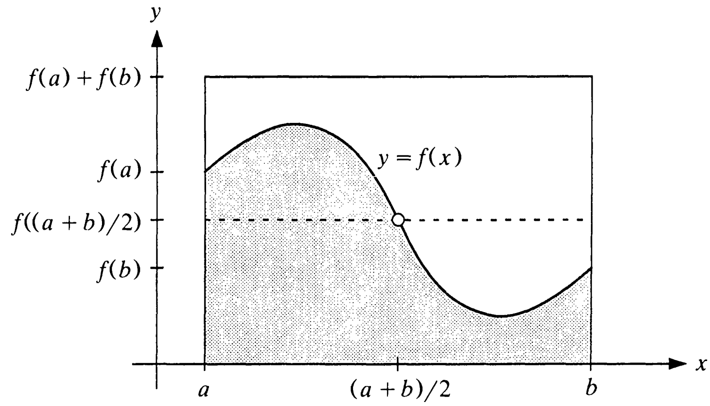

Figure 3
A "proof without words" of the theorem.

The reader is encouraged to try the following exercises, both by the preceding method and by a standard method for comparison. Answers are given below.*

1. $\int_{-1}^{1} \arctan(e^x)\,dx$

2. $\int_{-1}^{1} \arccos(x^3)\,dx$

3. $\int_0^2 \frac{dx}{x+\sqrt{x^2-2x+2}}$

4. $\int_0^2 \sqrt{x^2-x+1} - \sqrt{x^2-3x+3}\ \mathrm{dx}$

5. $\int_0^4 \frac{dx}{4+2^x}$

6. $\int_0^{2\pi} \frac{dx}{1+e^{\sin x}}$

Designing a Rose Cutter

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Most students of mathematics appreciate demonstrations of the relevance of mathematics in their chosen fields of study, and engineering students essentially insist on them. While almost all ordinary differential equations books include a brief discussion of first-order differential equations of the form $y' = g(y/x)$, few provide an example from engineering. I offer one here.

The problem is to design blades for a pair of pruning shears, consisting of one straight blade and one curved blade, with the specification that the angle between the two blades be constant regardless of how far the jaws are open.

Figure 1 shows the blades in the open position. The edge of the straight blade is the segment OB, with the hinge point at the origin. We assume that $\theta_0 = \pi/3$ radians and that each blade measures 5 cm from the hinge point to the tip, so

*Answers: **1.** $\pi/2$; **2.** π; **3.** 1; **4.** 0; **5.** 1/2; **6.** π.