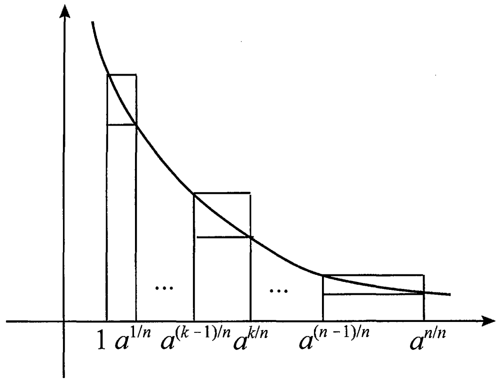

Figure 1.

Thus,

$$n(1-a^{-1/n}) < \int_1^a \frac{1}{x}\,dx < n(a^{1/n}-1).$$

Subtracting,

$$0 < \ln a - n(1-a^{-1/n}) < n(a^{1/n}-1) - n(1-a^{-1/n}) < n\left(\frac{a-1}{n}\right)^2,$$

where we used (1) and the assumption that $a > 1$ in the last inequality. Similarly,

$$0 < n(a^{1/n}-1) - \ln a < n\left(\frac{a-1}{n}\right)^2.$$

Taking limits, if $a > 0$,

$$\lim_{n\to\infty} n(1-a^{-1/n}) = \lim_{n\to\infty} n(a^{1/n}-1) = \ln a.$$

Exercise: Show that $\int_1^e \ln x\,dx = 1$ using Riemann sums.

An Application of L'Hôpital's Rule

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Recently, while teaching a course in calculus, I asked my students to do the following exercise:

Let $f(x)$ be differentiable on an interval (a, ∞) and $\lim_{x\to\infty}(f'(x) + f(x)) = L$ (L may be infinite). Prove $\lim_{x\to\infty} f(x) = L$.

A solution given by a student raised my interest: let $g(x) = e^x f(x)$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{g(x)}{e^x} = \lim_{x\to\infty} \frac{g'(x)}{e^x} = \lim_{x\to\infty} (f(x) + f'(x)) = L.$$

Is this proof right or wrong? Clearly it is wrong in the sense that an application of the ∞/∞ version of L'Hôpital's Rule was intended but, in fact, we do not know that $\lim_{x\to\infty} g(x) = \infty$.

One way to give a complete proof of the exercise using the student's approach is to handle separately the three cases in which $\lim_{x\to\infty} f(x)$ has a non-zero value (including $\pm\infty$), has a value of 0, or fails to exist. In the first case, $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} e^x f(x) = \pm\infty$ and L'Hôpital's Rule can be applied just as the student stated. If $\lim_{x\to\infty} f(x) = 0$ we cannot use L'Hôpital's Rule because $\lim_{x\to\infty} g(x)/e^x$ need not have the ∞/∞ form. However, by the Mean Value Theorem, we know that there exists, for every positive integer $n > a$, $x_n \in [n, n+1]$ with $f'(x_n) = f(n+1) - f(n)$. Then $\lim_{n\to\infty} f'(x_n) = 0$. Moreover, since $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}(f(x) + f'(x)) = L$. So $L = 0$.

For the final case, assuming $\lim_{x\to\infty} f(x)$ fails to exist (and is not $\pm\infty$), we shall obtain a contradiction. Let E denote the necessarily infinite and unbounded set of all $x \in (a, \infty)$ where f has a local extreme value. By Rolle's Theorem we have $f'(x) = 0$ for all $x \in E$. Now let $\{x_n\}$ be any sequence in (a, ∞) with $\lim_{n\to\infty} x_n = \infty$. Since the numbers x_n are not generally in E, choose two sequences $\{y_n\}$ and $\{z_n\}$ in E tending to ∞ and satisfying, for each n, $y_n \le x_n \le z_n$ and either $f(y_n) \le f(x_n) \le f(z_n)$ or $f(z_n) \le f(x_n) \le f(y_n)$. Then from $\lim_{x\to\infty}(f(x) + f'(x)) = L$ we have $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n) = L$. It follows that $\lim_{n\to\infty} f(x_n) = L$ and, hence, $\lim_{x\to\infty} f(x) = L$. This is a contradiction.

Now it seems that the discussion should end here. But, in fact, there is a generalized L'Hôpital's Rule that can be applied to solve the exercise directly.

Generalized L'Hôpital's Rule: Let f and g be differentiable on (a, ∞) and assume $g'(x) \ne 0$ for $x \in (a, \infty)$. If $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} f(x)/g(x) = L$ (where L may be infinite), then $\lim_{x\to\infty} f'(x)/g'(x) = L$.

Note that we do not assume that $\lim_{x\to\infty} f(x) = \infty$.

Proof. We will assume that L is finite and leave the cases $L = \pm\infty$ to the interested reader. Since $\lim_{x\to\infty} f'(x)/g'(x) = L$, we know that for any $\varepsilon > 0$ there exists $N_1 > a$ such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{2}{9}\varepsilon, \quad x > N_1. \tag{1}$$

But from $\lim_{x\to\infty} g(x) = \infty$ we know that

$$\lim_{x\to\infty} \frac{g(y)}{g(x)} = \lim_{x\to\infty} \frac{f(y)}{g(x)} = 0$$

for any fixed $y \in (N_1, \infty)$. Then there exists $N > y$ such that

$$\left|\frac{g(y)}{g(x)}\right| < \frac{1}{2}, \quad \left|\frac{f(y)}{g(x)}\right| < \frac{\varepsilon}{3}, \quad \text{and} \quad \left|L \cdot \frac{g(y)}{g(x)}\right| > \frac{\varepsilon}{3} \tag{2}$$

hold for any $x > N$.

Since f and g are differentiable and $g'(x) \ne 0$ on (a, ∞), by the Cauchy Mean Value Theorem we know that for any $x > N$ there exists $\xi \in (y, x)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$

On the other hand,

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)}}{1 - \frac{g(y)}{g(x)}}.$$

Thus

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)} \cdot \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)},$$

and then

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &= \left|\left(\frac{f'(\xi)}{g'(\xi)} - L\right) \cdot \left(1 - \frac{g(y)}{g(x)}\right) - L \cdot \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}\right| \\
&\le \left|\frac{f'(\xi)}{g'(\xi)} - L\right| \cdot \left(1 + \left|\frac{g(y)}{g(x)}\right|\right) + \left|L \cdot \frac{g(y)}{g(x)}\right| + \left|\frac{f(y)}{g(x)}\right| \\
&\le \frac{2}{9}\varepsilon \cdot \frac{3}{2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

where (1) and (2) are used. This means that $\lim_{x\to\infty} f(x)/g(x) = L$.

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The Attraction of Surfaces of Revolution

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In my lectures for the first-year calculus sequence, I state and solve physics problems. After the section on surface area, the following problem generated some interest:

Assuming an inverse square law of attraction, what is the force exerted by a massive surface of revolution on a point mass m located on the axis of symmetry?

An important special case is the attractive force of gravity exerted by a spherical shell on a point mass m. Since any line through the center is an axis of symmetry, m can be anywhere in space.

For the general case, here are some preliminary assumptions:

1. The surface of revolution is defined by a nonnegative function $f(x)$ on a closed interval $[a, b]$, such that f' exists on (a, b). The graph of f is revolved around the x-axis as in Figure 1.

2. The surface's mass is distributed evenly, in the sense that it has a constant "planar density," $d \ge 0$. The units on d might be kilograms per square meter, for example, to distinguish it from linear or spatial density.

3. The "inverse square law" refers to a force exerted on a point mass m by another point mass M separated by distance $r > 0$. Then the magnitude of the force is $GmMr^{-2}$, for a positive constant G. M and m will be assumed nonnegative, and the direction of the force on m is toward M.

4. To simplify calculation, the point mass m can be assumed to be at the origin, by translating f if necessary.

