CLASSROOM CAPSULES

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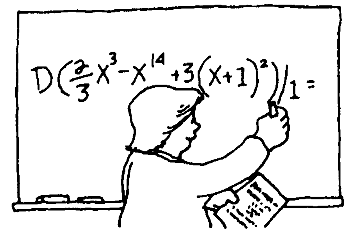

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

Interactive Teaching Aids for Multivariable Calculus

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One of the biggest challenges in a multivariable calculus course is helping students visualize concepts. Unlike our colleagues in physics, mathematicians have few visual aids available. Here we would like to describe four home-made visual aids that we have found to be effective in presenting such concepts as lines and planes in three-space, directional derivatives, gradients, parametric motion and differentials.

Planes with Normal Vectors

When trying to explain how two planes intersect or how a normal vector describes a plane, instructors often use an open book to illustrate the two planes and a pen to describe a normal vector. We have found that two pieces of plexiglas, approximately 16×16 inches, make a much better teaching aid. We took two pieces of plexiglas, cut each of them from the center of one edge to the center of the plexiglas, then mated the slits and joined them with duct tape as shown in Figure 1. A normal vector can easily

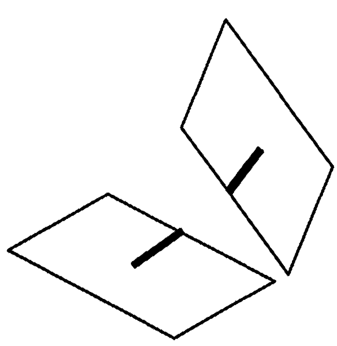

Figure 1. How to construct two planes

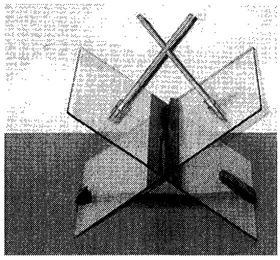

Figure 2. Plexiglas with Normal Vectors (Actual)

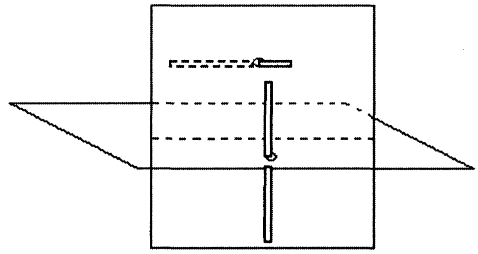

Figure 3. Planes with Normal Vectors (Schematic)

be described by putting a small hole in two of the sheets and inserting a wooden dowel (Figures 2 and 3). The plexiglas works better than wood or cardboard since it allows students to see through the planes. Using this visual aid the students are able to see that the intersection of two non-parallel planes is a line. They are also able to visualize how the two angles between the normal vectors of the two planes are equivalent to the two angles between the planes. In particular, they can readily see that two planes are orthogonal if and only if their normals are orthogonal. We can use the same device to help students determine how to find the distances between a point and a plane and between a line and a plane.

Modeling Clay

Another idea that has worked surprisingly well has been the use of modeling clay. Surfaces made from play-doh or modeling clay can be used to show the relationship between (x, y) and $f(x, y)$. Many students struggle with this basic concept until they can see it with their own eyes. To help them visualize the relationship, draw an x and y axis and a point, (x, y), on a sheet of paper or a piece of poster board. Place your mound of clay, the surface, on top of the axis and mark $f(x, y)$. Then a student can see what the $f(x, y)$ is, given x and y. This may require lifting the clay and replacing it in the same place several times so that the students can see the relationship between the point (x, y) on the paper and the value of $f(x, y)$ on the clay. This seems like a simple concept but if a student does not get it right away, and more than a few do not, trying to communicate ideas about multivariable calculus is very difficult.

Another benefit of using clay in a multivariable calculus course comes from cutting through a surface (again, a mound of clay) with a straightedge and showing how the partial derivative with respect to x gives the slope or the rate of change in the x direction. The same can be done in the y direction (Figure 4). Once the students understand this idea, you can then easily show them the concept of directional derivatives by placing a pencil on the paper/poster board in the direction of some arbitrary vector $< a, b >$. Then, cut the clay in this direction and use an arrow or pencil to show the slope at some point (x, y) (Figure 5). The nice thing about it is that you can put the clay back together again, pick a different vector, $< c, d >$, using the same point, (x, y), and show the students that the directional derivative is not the same.

Modeling clay can be used to clarify several other concepts. Level curves are easy to explain by horizontally slicing your surface or mound of clay at several elevations. After you remove the portion that you just sliced, you can show the students that every point along the resulting curve has the same $f(x, y)$ value. The concept of a gradient

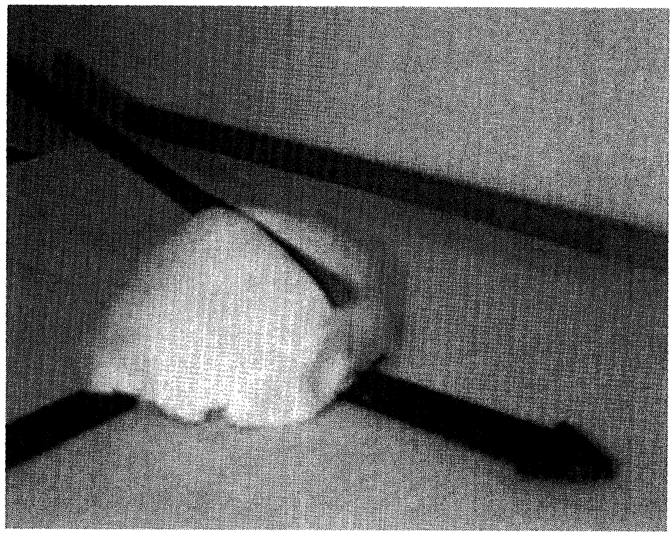

Figure 4. Modeling clay (Partial derivative)

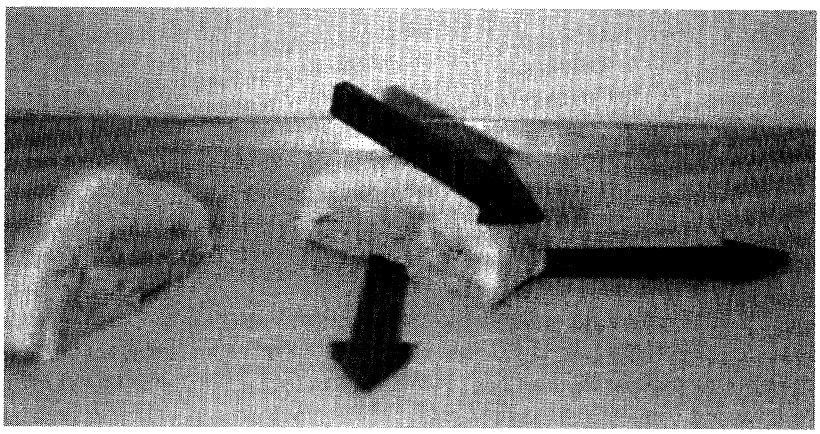

Figure 5. Clay (Directional derivative)

becomes clearer by doing the following: have your students calculate the gradient at a point such as $(1, 0)$ for a simple function like $z = \sqrt{x^2 + y^2}$, $z \geq 0$. Then plot the gradient on a piece of paper. Extend the gradient with hyphens or dots beyond its normal size so that the students can see its direction when you superimpose the clay. Form a small hill with the clay, representing the function mentioned above. Insert a toothpick vertically into the clay so it protrudes from the surface above the point $(1, 0)$. Now you are ready to show how the gradient points uphill in the direction of steepest ascent and how the rate of change is less in other directions.

Parametric Motion

Another teaching aid that has helped students to visualize concepts better has been a model of a particle moving in space (Figure 6). The model is made of a section of

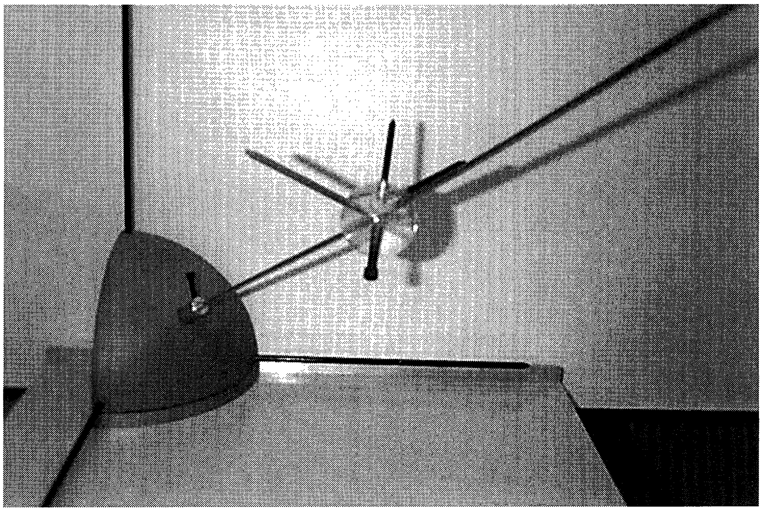

Figure 6. Parametric Motion

wood (any size or shape will do) with dowels (or pencils) exiting at right angles to represent the x, y and z axes. The x, y, and z axes give the students a fixed reference for orientation so that you can move the visual aid to different places in the classroom so students can see better what is being described. Then, there is a curved wire coming out of the model to represent the path of a particle in space. The wire must be rigid enough to maintain its form and to hold the weight of a small ball (particle), but flexible enough to be bent into different shapes; the wire from most clothes hangers meets these specifications. The students gain a better understanding of parametric equations if they can see the path that the particle follows and the location of the particle at different times.

We put a small ball on the wire so that we can move the ball forward and backward along it to represent the particle's position at different times. Additionally, the ball helps students to see the difference between the tangent, normal, and binormal vectors. We have put three small colored pegs into the ball so that one peg is pointing in the direction of movement (tangent vector), another is perpendicular to the direction of movement (normal vector), and the third is perpendicular to the previous two (binormal vector). Golf tees work well for the colored pegs. By fixing the ball at a specific point and allowing it to rotate, the students can see that while there is only one unit tangent vector, there are infinitely many unit vectors orthogonal to the tangent vector; the normal and binormal vectors are two of these.

Differentials and Surface Area

Many students have particular trouble visualizing linear approximations and differentials. A simple three-dimensional model helps them to get a picture in their mind (Figure 7). Our model contains three different pieces of plexiglas glued together at a particular point. The top piece of plexiglas, which is curved, represents a function of two variables (a surface). The middle piece represents the planar approximation at a given point. Using this device, the student can easily see the difference between the tangent plane approximation at a given point and the function. In Figure 7, the point of tangency is at the front left corner. The bottom piece of plexiglas, which is parallel

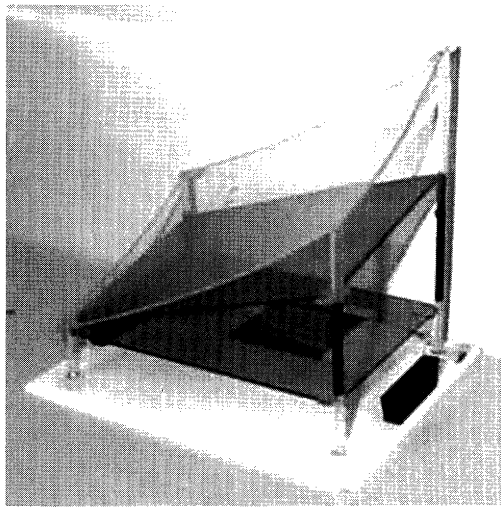

Figure 7. Differentials

to the ground, represents the change of x and the change of y of the function as you move from the front left point of the function. By looking at this model, students can see the physical difference between dz and Δz. They begin to understand that some of the change in z is attributable to the change in x and some of it is attributable to the change in y. Once they can see and grasp these ideas with functions of two variables, they can more easily move to functions of more than two variables.

The model is not difficult to construct from pre-cut plexiglas, available at most home improvement centers. We used sheets that were 8″ by 10″. The biggest challenge in constructing the model was to create a curved surface, but putting a piece of plexiglas in an oven at 500 degrees for 3–4 minutes makes it just soft enough to bend with a little force. After you bend the plexiglas to a shape you are happy with, put it in cold water to cool it and fix the shape.

Because all of the aids mentioned in this article are small, they are most effective in classes of less than fifty students. For large classes, multiple copies can be constructed (these devices are inexpensive and easy to produce). Additional copies would allow teaching assistants to use them in smaller groups to reinforce or clarify difficult concepts. The above are just a few of the ideas we used to help students visualize multivariable concepts. Students repeatedly commented on how useful they were in helping them understand the material rather than just memorizing it. Additionally, instructors enjoyed bringing innovative yet simple "toys" into the classroom.

Integration from First Principles

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The following approach seems to involve an extension of the standard argument for finding from first principles the value of a definite integral. Throughout we suppose that a and b are real numbers satisfying $0 < a < b$ and that

$$\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$