The Cantor Set Contains $\frac{1}{4}$? *Really?*

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The authors have been teaching undergraduate real analysis this year from Marsden's wonderful book [**1**]. Marsden introduces the Cantor set and many of its properties in an exercise. We find it beneficial for undergraduates to study the Cantor set because it is a good source of counterexamples for what might otherwise seem reasonable conjectures. Also, while the Cantor set is complicated, its definition is fairly simple and easy to understand. As do most undergraduate texts, Marsden uses the middle-thirds definition of the Cantor set.

Definition. Let $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and construct F_{k+1} from F_k by deleting the middle third of each interval in F_k. Then the *Cantor set* $C = \bigcap_{n=1}^{\infty} F_n$.

After our students had been thinking about the Cantor set for a couple of weeks, we heard them make the following sort of comment: "In the limit, isn't it true that all points in the Cantor set are endpoints of those intervals?" ... and *then* ... "Our intuition is probably wrong, so there have to be other points in the Cantor set. But they must be some kind of really weird irrational points."

It's easy to see why beginning analysis students would draw such conclusions. After all, if you get rid of the middle third of *every* interval, you should not have *any* intervals in the Cantor set, and the only points left are those of the form $\frac{m}{3^n}$, right?

Wrong.

Analyzing the situation. We immediately knew that this intuition was incorrect—after all, the Cantor set is uncountable, while the set of "the endpoints" is countable. When informed of this, students might naturally ask "So, can you name some point in C other than the endpoints?" As do most mathematicians, when thinking analytically about the Cantor set we use the fact that C is equivalent to the set of points in the interval $[0, 1]$ whose ternary expansion can be written using only the digits 0 and 2. (This is given in most graduate texts such as [**2**] and [**3**].) Using this, it is immediate that C is uncountable, and relatively easy to see that there are many rationals other than "the endpoints" in the Cantor set. In particular, the ternary expansion of $\frac{1}{4}$ is .0202020202.... We believe that this fact is important! First of all, it is surprising to beginning analysis students (at least it was to our students); secondly, it enhances their intuition and understanding of the Cantor set. Finally, it's just plain cool!

Who knows that $\frac{1}{4}$ is in the Cantor Set? We began to wonder:

- Do most mathematicians know this?
- If so, when do they learn it?
- Do many undergraduates learn this fact?
- How can we convey this to our students without resorting to the not-so-intuitive ternary expansion?

An informal survey (sample size 12) caused us to conclude that most mathematicians either did not know or did not remember this fact (9/12). (Be honest. Did *you* know it?) However, most mathematicians can prove that $\frac{1}{4}$ is in the Cantor set within seconds using the ternary expansion. Generally (12/12) mathematicians discover this fact in graduate school or later.

We surveyed approximately twenty undergraduate analysis texts and found that only one (very recent) text [**4**] mentioned that $\frac{1}{4}$ is in the Cantor set, and only after having first introduced the ternary expansion. But we can avoid wrangling with the complexities of the ternary expansion!

How we like to present this to our students. We can convince our students that $\frac{1}{4}$ could be in C, by illustrating successive iterations of removing middle-thirds, as in Figure 1:

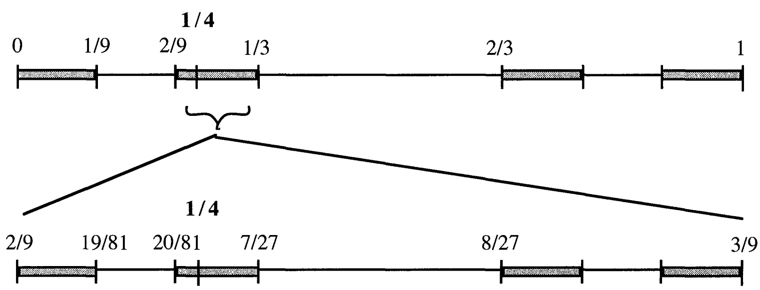

Figure 1.

In the next three iterations, $\frac{1}{4}$ appears in $[\frac{182}{729}, \frac{61}{243}]$, $[\frac{1640}{6561}, \frac{547}{2187}]$, and $[\frac{14762}{59049}, \frac{4921}{19683}]$—exactly one-fourth of the way from the left endpoint to the right endpoint.

What remains is an analytic proof. We use the elegant expansion $\frac{1}{4} = \frac{1}{3} - \frac{1}{3^2} + \frac{1}{3^3} - \frac{1}{3^4} + \ldots$. First, it is clear that the right-hand side of the equality represents an element in the Cantor set, for each partial sum corresponds to travelling to a successive interval endpoint (all of which are in each set in the infinite intersection that is the Cantor set). Figure 2 shows a graphical interpretation.

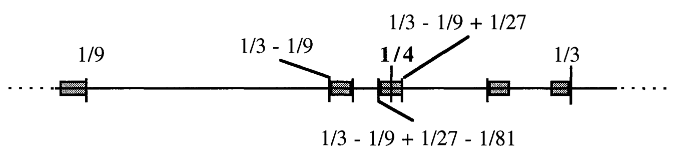

Figure 2.

Then, we can see that the series converges to $\frac{1}{4}$ by examining the geometric series formula $\frac{1}{4} = \frac{A}{(1-r)}$, where in this case $A = \frac{1}{3}$ and $r = \frac{-1}{3}$.

This idea may be introduced by asking students whether there are rationals in the Cantor set other than those of the form $\frac{m}{3^n}$, and later suggesting that perhaps there *are* other rationals, even obvious ones. That leads to showing that $\frac{1}{4}$ is in C. We believe that this beautiful, simple, and somewhat suprising fact should be a standard example in any undergraduate analysis course.

References

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2. H. L. Royden, *Real Analysis*, Macmillan, 1988.
3. W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.
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