Table 5

$\int(\ln x)^2\,dx = x[(\ln x)^2 - 2\ln x + 2] + C$

uv	$uv' + u'v$
$(\ln x)^2(x)$	$(\ln x)^2 + 2\ln x$
$-(2\ln x)(x)$	$-2\ln x - 2$
$+2x$	$+2$

Another textbook favorite is $\int \sin\ln x\,dx$. The original method runs into the same problem as the preceding example; but the variant, Table 6, is smooth and in fact is isomorphic to Table 3.

Table 6

$\int \sin\ln x\,dx = \frac{1}{2}x(\sin\ln x - \cos\ln x) + C$

uv	$uv' + u'v$
$(\sin\ln x)(x)$	$\sin\ln x + \cos\ln x$
$-(\cos\ln x)(x)$	$-\cos\ln x$
	$+\sin\ln x$

Four Crotchets on Elementary Integration

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All of the results below are well known to too few people.

1. Integral of Exponential Times Polynomial. Most students, when confronted with

$$\int_0^2 e^{3x}(x^3 + 6x^2 + 11x + 6)\,dx,$$

write the integral as the sum of four integrals and evaluate them separately, using integration by parts six times altogether. They haven't learned that a polynomial is a single function, so that only three successive integrations by parts are needed. However, explicit integration by parts can be avoided by use of a single formula, which is more useful than many of the integration formulas customarily memorized by students.

Let P be a polynomial, and m a nonzero constant. Then

$$\int e^{mx}P(x)\,dx = \frac{e^{mx}}{m}\left(P(x) - \frac{P'(x)}{m} + \frac{P''(x)}{m^2} - \frac{P'''(x)}{m^3} + \cdots\right) + c.$$

(This is essentially exercise 15 on p. 225 of Courant [2].) The proof is by repeated use of the recurrence formula

$$\int e^{mx}P(x)\,dx = \frac{e^{mx}}{m}P(x) - \int e^{mx}\frac{P'(x)}{m}\,dx,$$

obtained by a single integration by parts.

Similar but slightly more complicated formulas may be obtained in the same way for $\int(\sin(mx))P(x)\,dx$ and $\int(\cos(mx))P(x)\,dx$.

Many students haven't learned how to use integration by parts with definite integrals. Thus, $\int_0^2 e^x x\,dx$ is often "evaluated" as

$$e^x x - \int_0^2 e^x\,dx = e^x x - e^2 + 1,$$

which incorrectly depends on x. A better formulation is

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_{x=a}^b - \int_a^b u'(x)v(x)\,dx,$$

or

$$\int_a^b u(x)v'(x)\,dx = \left[u(x)v(x) - \int u'(x)v(x)\,dx\right]_{x=a}^b.$$

The second form is useful when the separate terms in the first form are undefined, as often occurs in improper integrals.

2. Using limits of integration. The notation $[F(x)]_{x=a}^b$ for $F(b)-F(a)$ avoids the ambiguities associated with the commonly used $F(x)|_a^b$ in two ways: (1) it specifies the letter to be substituted **for**, here x; and (2) it specifies the scope of the formula to be substituted **into**, here $F(x)$. Without these specifications, it is uncertain what $x - 2y + xy^2|_3^5$ means. The notation $[F(x)]_{x=a}^b$ can be used more generally to mean $\lim_{x\nearrow b} F(x) - \lim_{x\searrow a} F(x)$, where the limits are taken from inside the interval (a,b), as with improper integrals. Use of slanted arrows for one-sided limits avoids the confusion caused by nonnumbers like $-(2^+)$ and $(-2)^+$.

3. Integration by substitution. Many students evaluate

$$\int_0^4 12(2x-3)^5\,dx$$

by first multiplying out. (They don't know the binomial theorem.) More enterprising students make the substitution $u := 2x-3$, evaluate an indefinite integral, and then substitute back before using the limits of integration:

$$\int 12(2x-3)^5\,dx = \int 6u^5\,du = u^6 + c = (2x-3)^6 + c.$$

However, it is usually simpler to find the appropriate limits of integration in terms of u (when t goes from 0 to 4, then u goes from -3 to 5) instead of substituting back. But many students do not change the limits of integration properly, and so write

$$u^6|_0^4 = 4096 \text{ instead of } [u^6]_{u=-3}^5 = 14896.$$

(Note the use of subscript u as a reminder that the limits of integration are now for u.)

4. Differential equations with initial conditions. To solve a first-order differential equation with initial condition, the procedure followed by nearly all textbooks is to find a general solution of the differential equation by means of indefinite integration, and then to find the constant of integrity by substitution. This interrupts the sequence of steps (equivalences) in the solution. It is better to use **definite** integration, as in the following example.

Express x in terms of t, where

$$dx/dt = 3x + 6 \text{ for all real } t, \text{ and } x = -5 \text{ when } t = 2.$$

The differential equation is both separable and linear. In general, such equations are best treated as linear equations (see [**1**], pp. 34–35, 39–40, etc.), since doing so avoids the error-prone process of getting rid of logarithms. After rewriting the equation in standard linear form and multiplying by the integrating factor $e^{\int(-3)\,dt}$ (choosing one antiderivative), we obtain $e^{-3t}\,dx/dt - 3e^{-3t}x = 6e^{-3t}$ for all real t, with initial condition. Now we integrate with respect to t from 2 to t, using the corresponding limits -5 and x for x, and obtain

$$[e^{-3t}x]_{(t,x)=(2,-5)}^{(t,x)} = [-2e^{-3t}]_{t=2}^{t} \quad \text{for all real } t$$

$$\Leftrightarrow e^{-3t}x + 5e^{-6} = -2e^{-3t} + 2e^{-6} \quad \text{for all real } t$$

$$\Leftrightarrow x = -2 - 3e^{3t-6} \quad \text{for all real } t.$$

In the conventional method, after integrating indefinitely and solving for x in terms of t, we obtain

$$x = -2 + ce^{3t} \quad \text{for all } t.$$

Use of the initial condition then requires "unsolving" the equation for c.

If, however, we solve the equation as a separable equation, we obtain

$$\frac{dx}{x+2} = 3\,dt \text{ with initial conditions}$$

$$\Leftrightarrow [\ln(x+2)]_{x=-5}^{x} = [3t]_{t=2}^{t} \quad \text{for all } t$$

$$\Leftrightarrow \ln(x+2) - \ln(-3) = 3t - 6 \quad \text{for all } t(??!!)$$

$$\Leftrightarrow x + 2 = -3e^{3t-6} \quad \text{for all } t \text{ (tricky!)}.$$

The antiderivative for the left side (without constant) is often written using absolute values as

$$\ln|x+2|,$$

which, however, does not indicate that, because of the initial condition, $x+2$ must have the same sign as -3. In general, without absolute values,

$$\int_a^b \frac{1}{x}\,dx = \ln\frac{b}{a}, \quad \textbf{provided that } ab > 0.$$

References

1. Ralph Palmer Agnew, *Differential Equations*, McGraw-Hill, New York and London, 1942.
2. R. Courant, *Differential and Integral Calculus*, Vol. 1, 2nd ed., translated by E. J. McShane, Blackie and Son, London and Glasgow, 1937.

A Shortcut in Partial Fractions

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The method of partial fractions is the basic technique for preparing rational functions for integration. It is also a useful tool for finding inverse Laplace transforms. This method enables us to write a rational function as a sum of simpler quotients that can be integrated directly or transformed easily by the inverse Laplace operator.

The basic technique to find partial fractions for a rational function is based on the method of undetermined coefficients. However, the computation involved in this method is often tedious. The following is a simple shortcut to expanding certain rational functions in partial fractions. We believe it is worthwhile to include this method in the texts.

Shortcut. Let $p(x)$ be a function and a, b distinct scalars. Then

$$\frac{1}{(p(x)+a)(p(x)+b)} = \left(\frac{1}{p(x)+a} - \frac{1}{p(x)+b}\right)\frac{1}{b-a}.$$

This is a special case of a general algebraic identity, and it is really useful. Let us look at some applications.

Example 1.

$$\frac{x}{(x^2+1)(x^2+4)} = \frac{x}{4-1}\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right) = \frac{1}{3}\left(\frac{x}{x^2+1}\right) - \frac{1}{3}\left(\frac{x}{x^2+4}\right).$$