A Pretrigonometry Proof of the Reflection Property of the Ellipse

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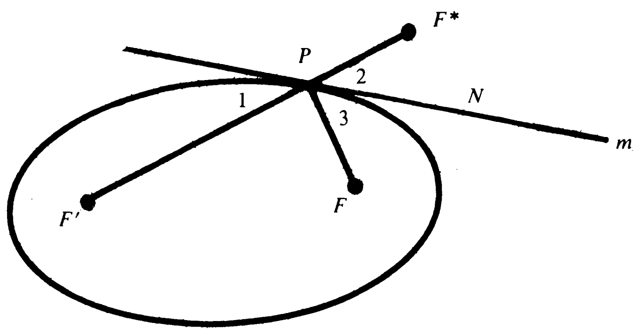
In any ellipse, the segments from the foci to any point on the ellipse make equal angles with the tangent. This is equivalent to the reflecting property applied in whispering galleries. The reflecting property is often proved using equations of lines, eccentricity, trigonometry, and/or calculus. All of that machinery disguises the important ideas, namely that this property is fundamentally geometric, not algebraic. To use coordinates distorts one's understanding.



Suppose line m is tangent to the above ellipse at P, and let $F'P + FP = 2a$. Since m is a tangent line, any point N on m other than P lies outside the ellipse; so any other point N satisfies

$$F'N + FN > 2a.$$

Let F^* be the reflection image of F over line m. Since reflections preserve distance, $FP = F^*P$ and $FN = F^*N$. Since they preserve angle measure, angles 2 and 3 have the same measure. We now show that P must be on $F'F^*$. We do this by showing that the distance from F' to F^* is minimized by going through P. Putting all the above together, we see that for all $N \neq P$:

$$F'P + F^*P = F'P + FP = 2a < F'N + FN = F'N + F^*N.$$

Thus, angles 1 and 2 are vertical angles and have the same measure. So angles 1 and 3 have the same measure, which was to be proved.

Numerical Integration via Integration by Parts

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In this note we illustrate how integration by parts can be used to obtain the familiar rectangular, trapezoidal, midpoint, and Simpson approximations of integrals. Our approach can serve to further enrich students' appreciation of the relationships among these numerical approximations.

Suppose $P(x)$ is a polynomial (to be determined), and $f(x)$ is continuously differentiable on $[a, b]$. From integration by parts,

$$\int P'f\,dx = Pf - \int Pf'\,dx. \tag{1}$$

If (1) is to provide an approximation to $\int f\,dx$, we must require $P'(x) = 1$. So assume $P(x) = x + B$.

Suppose we let $P_l(x) = x + B_l$ for $x \in [a,(a+b)/2]$, and $P_r(x) = x + B_r$ for $x \in [(a+b)/2, b]$. Then

$$\int_a^b f\,dx = \int_a^{(a+b)/2} P_l' f\,dx + \int_{(a+b)/2}^b P_r' f\,dx,$$

and by (1) applied to $[a,(a+b)/2]$ and $[(a+b)/2, b]$:

$$\begin{aligned}\int_a^b f\,dx = &-(a+B_l)f(a) + (B_l - B_r)f\left(\frac{a+b}{2}\right) + (b+B_r)f(b)\\ &-\int_a^{(a+b)/2} P_l f'\,dx - \int_{(a+b)/2}^b P_r f'\,dx.\end{aligned} \tag{2}$$

The two integrals on the right-hand side of (2) are the errors induced by approximating $\int_a^b f\,dx$ with $-(a+B_l)f(a) + (B_l - B_r)f\left(\frac{a+b}{2}\right) + (b+B_r)f(b)$. Note that if P_l and P_r are of constant sign on the appropriate intervals (e.g., P_l nonnegative), then

$$\min_{[a,(a+b)/2]} f' \cdot \int_a^{(a+b)/2} P_l\,dx \le \int_a^{(a+b)/2} P_l f'\,dx \le \max_{[a,(a+b)/2]} f' \cdot \int_a^{(a+b)/2} P_l\,dx,$$

and hence (by the Intermediate Value theorem), there is some $c \in (a,(a+b)/2)$ such that

$$\int_a^{(a+b)/2} P_l f'\,dx = f'(c)\int_a^{(a+b)/2} P_l\,dx.$$

Using (2), we can select B_l and B_r to obtain the particular approximations desired.

<u>Rectangular</u>. $\int_a^b f\,dx \sim (b-a)f(a)$ suggests $B_l = -b = B_r$. These values reduce (2) to

$$\begin{aligned}\int_a^b f\,dx &= (b-a)f(a) + \int_a^b (b-x)f'\,dx\\ &= (b-a)f(a) + f'(c_1)\int_a^b (b-x)\,dx \qquad c_1 \in (a,b).\end{aligned}$$

Thus,

$$\int_a^b f\,dx = (b-a)f(a) + \frac{(b-a)^2}{2}f'(c_1) \quad \text{for some } c_1 \in (a,b). \tag{R_l}$$

Similarly, $\int_a^b f\,dx \sim (b-a)f(b)$ suggests $B_l = -a = B_r$, for which (2) becomes

$$\begin{aligned}\int_a^b f\,dx &= (b-a)f(b) + \int_a^b (a-x)f'\,dx\\ &= (b-a)f(b) + f'(c_2)\int_a^b (a-x)\,dx \qquad c_2 \in (a,b).\end{aligned}$$

Hence,

$$\int_a^b f\,dx = (b-a)f(b) - \frac{(b-a)^2}{2}f'(c_2) \quad \text{for some } c_2 \in (a,b). \tag{R_r}$$

Trapezoid. $\int_a^b f\,dx \sim (b-a)\left[\frac{f(a)+f(b)}{2}\right]$ suggests $-a-B_l=(b-a)/2=b+B_r$ and $B_l-B_r=0$. Substituting $B_l=-(a+b)/2=B_r$ into (2) gives

$$\int_a^b f\,dx=\left(\frac{b-a}{2}\right)[f(a)+f(b)]+\int_a^{(a+b)/2}\left(\frac{a+b}{2}-x\right)f'\,dx$$

$$+\int_{(a+b)/2}^b\left(\frac{a+b}{2}-x\right)f'\,dx$$

$$=\left(\frac{b-a}{2}\right)[f(a)+f(b)]+f'(x_1)\frac{(b-a)^2}{8}-f'(x_2)\frac{(b-a)^2}{8}$$

for some $x_1\in(a,(a+b)/2)$ and $x_2\in((a+b)/2,b)$. Therefore,

$$\int_a^b f\,dx=\left(\frac{b-a}{2}\right)[f(a)+f(b)]+\frac{(b-a)^2}{8}[f'(x_1)-f'(x_2)]$$

$$\text{for } a<x_1<\frac{a+b}{2}<x_2<b. \quad \text{(T)}$$

Midpoint. $\int_a^b f\,dx \sim (b-a)f\left(\frac{a+b}{2}\right)$ suggests $-a-B_l=0=b+B_r$ and $B_l-B_r=b-a$. Substituting $B_l=-a$ and $B_r=-b$ into (2) yields

$$\int_a^b f\,dx=(b-a)f\left(\frac{a+b}{2}\right)+\int_a^{(a+b)/2}(a-x)f'\,dx+\int_{(a+b)/2}^b(b-x)f'\,dx,$$

which gives

$$\int_a^b f\,dx=(b-a)f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{8}[f'(y_1)-f'(y_2)]$$

$$\text{for } a<y_1<\frac{a+b}{2}<y_2<b. \quad \text{(M)}$$

Simpson. $\int_a^b f\,dx \sim \left(\frac{b-a}{6}\right)[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)]$ suggests $-a-B_l=(b-a)/6=b+B_r$ and $B_l-B_r=4(b-a)/6$. Substituting $B_l=-(5a+b)/6$ and $B_r=-(a+5b)/6$ into (2), we obtain

$$\int_a^b f\,dx=\left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]+\int_a^{(a+b)/2}\left(\frac{5a+b}{6}-x\right)f'\,dx$$

$$+\int_{(a+b)/2}^b\left(\frac{a+5b}{6}-x\right)f'\,dx.$$

Hence,

$$\int_a^b f\,dx=\left(\frac{b-a}{6}\right)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$
$$+\frac{(b-a)^2}{72}[f'(z_1)-4f'(z_2)+4f'(z_3)-f'(z_4)] \quad \text{(S)}$$

for $a<z_1<(5a+b)/6<z_2<(a+b)/2<z_3<(a+5b)/6<z_4<b$.

It should be noted that with additional smoothness assumptions on f, we can obtain higher-order error estimates and compare the attractiveness of one method over another.

Let's consider another application of our basic idea: estimating the integral of f in terms of a few discrete values of f, and an integral whose integrand involves f' and a polynomial. Use $B_l = -b = B_r$ in (2) to obtain

$$\int_a^b g\,dx = (b-a)g(a) + \int_a^b (b-x)g'\,dx, \tag{R}$$

and then replace g with $(b-x)g'$. This yields

$$\begin{aligned}\int_a^b (b-x)g'\,dx &= (b-a)(b-a)g'(a) + \int_a^b (b-x)[(b-x)g']'\,dx \\ &= (b-a)^2 g'(a) + \int_a^b (b-x)^2 g''\,dx - \int_a^b (b-x)g'\,dx,\end{aligned}$$

or,

$$\int_a^b (b-x)g'\,dx = \frac{1}{2}(b-a)^2 g'(a) + \frac{1}{2}\int_a^b (b-x)^2 g''\,dx.$$

Therefore, upon substitution into (R),

$$\int_a^b g\,dx = (b-a)g(a) + \frac{1}{2}(b-a)^2 g'(a) + \frac{1}{2}\int_a^b (b-x)^2 g''\,dx.$$

Repeating this process, we eventually obtain

$$\begin{aligned}\int_a^b g\,dx = (b-a)g(a) + \frac{1}{2!}(b-a)^2 g'(a) + \cdots + \frac{1}{(n-1)!}(b-a)^{n-1} g^{(n-2)}(a) \\ + \frac{1}{(n-1)!}\int_a^b (b-x)^{n-1} g^{(n-1)}\,dx.\end{aligned}$$

Now replace $g(x)$ with $f'(x)$ and substitute x for b, and t for x. Then

$$\begin{aligned}f(x) - f(a) &= \int_a^x f'(t)\,dt \\ &= (x-a)f'(a) + \cdots + \frac{1}{(n-1)!}(x-a)^{n-1} f^{(n-1)}(a) \\ &\quad + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt,\end{aligned}$$

and we arrive at

$$\begin{aligned}f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} \\ + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt,\end{aligned}$$

Taylor's theorem with the integral form of the remainder.

Another seemingly frivolous (but useful in functional analysis) application of (R) is the following: Among all continuously differentiable functions f on $[0,1]$ with

$f(0) = 0$, is it true that

$$\left(\int_0^1 f\,dx\right)^2 \le K\int_0^1 |f'|^2\,dx$$

for some constant K?

Beginning with (R), we have $\int_0^1 f\,dx = \int_0^1 (1-x)f'\,dx$. Therefore,

$$\begin{aligned}\left(\int_0^1 f\,dx\right)^2 &= \left(\int_0^1 (1-x)f'\,dx\right)^2\\ &\le \left(\int_0^1 (1-x)|f'|\,dx\right)^2\\ &\le \int_0^1 (1-x)^2\,dx \cdot \int_0^1 |f'|^2\,dx\\ &= \frac{1}{3}\int_0^1 |f'|^2\,dx,\end{aligned}$$

where the second inequality is an application of Schwarz's inequality. Thus, our answer is affirmative, with $K = 1/3$. The interested reader might try to show that $K = 1/3$ is best possible by showing that there exists a continuously differentiable function f on [0,1] with $f(0) = 0$ and $(\int_0^1 f\,dx)^2 = (1/3)\int_0^1 |f'|^2\,dx$.

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Behold! The Pythagorean Theorem via Mean Proportions

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$$\frac{c+a}{b} = \frac{b}{c-a}$$

$$a^2 + b^2 = c^2$$