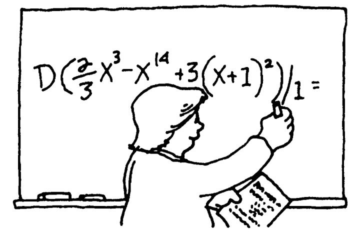
CLASSROOM CAPSULES

EDITOR

Thomas A. Farmer
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056-1641



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Maximizing the Arclength in the Cannonball Problem

Ze-Li Dou (zdou@gamma.is.tcu.edu) and Susan G. Staples (sstaples@gamma.is.tcu.edu), Texas Christian University, Forth Worth, TX 76129

Every calculus instructor is familiar with some version of the "cannonball problem" in which a cannonball is fired from ground level with an initial speed v and an initial angle of inclination θ, $0 \le \theta \le \pi/2$. Typical questions are:

1. For what length of time T is the cannonball in flight?
2. What is the horizontal range R covered by the cannonball?
3. For what angle θ is R maximized?

We suggest a follow-up problem, which may be used after the concept of arclength and techniques of integration are covered.

4. What angle θ maximizes the arclength of the trajectory?

We write, as usual,

$$x = vt\cos\theta \qquad \text{and} \qquad y = vt\sin\theta - \tfrac{1}{2}gt^2,$$

where g is the gravitational constant. Then the well-known answers to the first three questions are $T = 2v\sin\theta/g$, $R = v^2\sin 2\theta/g$, and $\theta = \pi/4$, respectively. The arclength of the trajectory is

$$L(\theta) = \int_0^T \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2v\sin\theta/g} \sqrt{(v\cos\theta)^2 + (gt - v\sin\theta)^2}\, dt.$$

Making the substitution $u = gt - v\sin\theta$, we obtain

$$L(\theta) = \frac{1}{g}\left(\int_{-v\sin\theta}^{v\sin\theta} \sqrt{u^2 + v^2\cos^2\theta}\, du\right).$$

After utilizing a standard trigonometric substitution, we see that, for $0 \le \theta < \pi/2$,

$$L(\theta) = \frac{1}{g}\left(v^2 \sin\theta + v^2 \frac{\cos^2\theta}{2} \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|\right),$$

while the case $\theta = \pi/2$, corresponding to shooting the cannonball straight up, leads to $L(\pi/2) = v^2/g$.

To compute the critical values, we take the derivative of $L(\theta)$ and find

$$L'(\theta) = \frac{1}{g}\left(2v^2 \cos\theta - v^2 \cos\theta \sin\theta \ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|\right).$$

The critical value for θ in $(0, \pi/2)$ is the angle θ_0 which satisfies

$$\sin\theta_0 \ln\left|\frac{1+\sin\theta_0}{1-\sin\theta_0}\right| = 2.$$

At the endpoints of the interval $[0, \pi/2]$, we see that $L(0) = 0$ and that $L(\pi/2)$ gives a local minimum. Using a graphing calculator, or by Newton's method, one easily obtains $\theta_0 = .985514738\ldots$ radians or about 56 degrees. It is interesting to notice that this θ_0, which maximizes the arclength, lies strictly between the range-maximizing angle $\pi/4$ and the height-maximizing angle $\pi/2$.

Pictures Suggest How to Improve Elementary Numerical Integration

Keith Kendig (kendig@math.csuohiu.edu), Cleveland State University, Cleveland, OH 44115

In a recent introductory numerical methods course, Maple helped students discover some substantial improvements to the trapezoidal and Simpson's methods. Students had learned to do numerical integration using rectangles and trapezoids, and were just starting to use Simpson's formula.

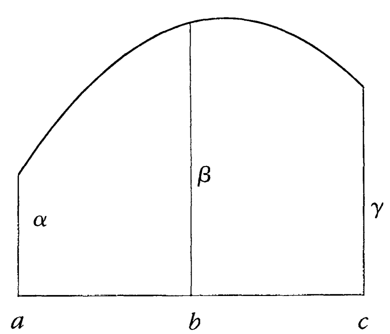


Figure 1

In Figure 1, the base is $(c-a)$, and $\frac{1}{6}(\alpha + 4\beta + \gamma)$ is an average altitude. The formula comes from passing a parabolic arc passing through three points; this arc