The Chair, the Area Rug, and the Astroid

Mark Schwartz, Ohio Wesleyan University, Delaware, OH 43015

The failing desk chair in my office was recently replaced by a shiny new ergonomic model. However, there was a problem. The new chair rolled much too freely across the tile floor. This put a strain on my knees, back, and dignity, as I struggled to keep the chair under control. I procured a rectangular area rug (actually a discarded piece of carpeting), which solved the chair problem but created another. The rug was so thick that the door would not close over it. Weighing various alternatives, I decided to implement a max/min solution. In particular, my strategy was to cut off the interfering corner, with the cut taken along a tangent line to the path of the door edge, such that the area of the removed piece was minimal. This brings us to the central problem.

Abstracted, the situation is illustrated in Figure 1. The edge of the door travels along the unit circle in the first quadrant. The corner of the carpet is at point (a, b). The problem is to determine the "cutting point" (u, v) on the quarter-circle such that the triangle formed by the tangent line to the circle at (u, v) and the lines $x = a$, $y = b$ has minimal area. Throughout the discussion, $v = \sqrt{1 - u^2}$.

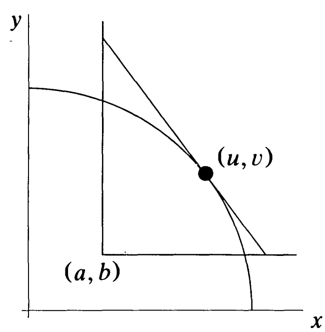

Figure 1

The tangent line to the quarter-circle at any point (u, v) is given by $y - v = (-u/v)(x - u)$. This intersects the lines $x = a$, $y = b$ at the points $(a, (1 - au)/v)$ and $((1 - bv)/u, b)$, respectively. Then, the area of the corresponding triangle is

$$A = \frac{1}{2}\left(\frac{1 - bv}{u} - a\right)\left(\frac{1 - au}{v} - b\right) = \frac{(1 - au - bv)^2}{2uv}.$$

Taking the natural log of both sides, differentiating with respect to u, and simplifying, we get

$$A' = A\left[\frac{2u^2 - au - 1 + bv}{uv^2(1 - au - bv)}\right].$$

Setting this equal to zero, we arrive at the condition

$$2u^2 - au - 1 = -bv = -b\sqrt{1 - u^2}. \tag{1}$$

Dropping the quest for an exact value for u, we work toward a geometric result. We find that (1) can be written $u^2 - au = 1 - u^2 - bv = v^2 - bv$, and so

$$\frac{u}{v} = \frac{v-b}{u-a}. \tag{2}$$

We see that u should be chosen so that the slope of the line segment from (a,b) to (u,v) is the same as the slope of the line segment from $(0,0)$ to (v,u). Equivalently, multiplying (2) by -1, u should be chosen so that the slope of the tangent line to the circle at (u,v) is the same as the slope of the line segment from (a,v) to (u,b). It is easy to verify that the point (u,v), which we will call the *cutting point*, minimizes the area of the "removed" triangle, and is unique.

To fully develop our geometric solution, consider the inverse problem. Given (u,v) as the cutting point, we determine the corresponding locus of rug corners (a,b). Referring to (1) or (2), this is easily seen to be the points on the line

$$y = \frac{u}{v}(x-u) + v. \tag{3}$$

Here the variables a and b have been replaced by x and y, respectively. Any point (a,b) on this line and inside the circle will have (u,v) as its cutting point. Figure 2 shows the attractive pattern of such lines resulting from several cutting points. Of interest is the envelope E, the curve that is tangent to all these lines. The method for computing E is standard (see any of the references, but **[1]** is particularly easy to read). Letting

$$f(x,y,u) = y - v - \frac{u}{v}(x-u) = y - \sqrt{1-u^2} - \frac{u}{\sqrt{1-u^2}}(x-u),$$

we just solve the equations $f(x,y,u)=0$ and $\frac{\partial f}{\partial u}(x,y,u)=0$ simultaneously for x and y in terms of u. Since

$$\frac{\partial f}{\partial u} = \frac{-2u^3+3u-x}{(1-u^2)^{3/2}}$$

we can readily obtain the result that E is given by the parametric equations

$$x = 3u - 2u^3$$
$$y = (2u^2+1)\sqrt{1-u^2}.$$

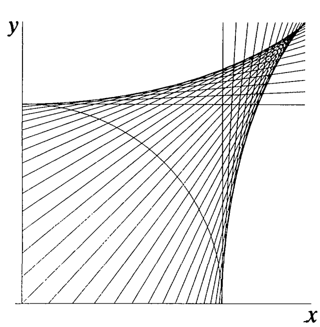

Figure 2

Taking $0 \leq u \leq 1$, we get the envelope E seen in Figure 2. This curve is well known, and we now reveal its identity.

The line (3) intersects the lines $y = x$ and $y = -x$ at the points $(u+v, u+v)$ and $(u-v, v-u)$, respectively. The distance between these points is $\sqrt{(2v)^2 + (2u)^2} = 2$. Thus, we see that E is the envelope of the family of line segments of length 2 with endpoints on the perpendicular lines $y = x$ and $y = -x$, as in Figure 3. This envelope is precisely an *astroid*, as detailed in any of the references. Our sought-after geometric solution to the rug cutting problem is apparent from this: The cutting point is the intersection of the unit circle with the line segment of length 2 passing through the point (a, b) having endpoints on the lines $y = x$ and $y = -x$. Figure 4 illustrates this.

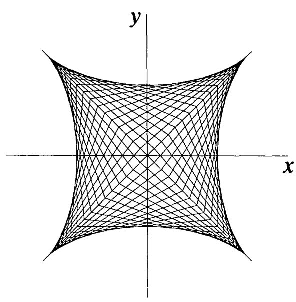

Figure 3

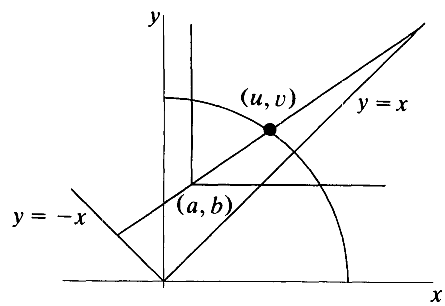

Figure 4

The references contain interesting and accessible examples, many of which would serve well as lab or independent projects for calculus students. For instance, in addition to showing that the astroid is the envelope of the family of line segments of a prescribed length with endpoints on perpendicular lines, the references demonstrate that the "doubled and rotated" astroid (the curve E above) is the evolute of the "unit" astroid in standard position. All offer much additional reading on evolutes and involutes as well.

References

1. V. C. Boltyanskii, *Envelopes* (*Popular Lectures in Mathematics*, vol. 12), Macmillan, New York, 1964.
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4. Robert C. Yates, *Curves and Their Properties* (*Classics in Mathematics Education*, vol. 4), National Council of Teachers of Mathematics, Reston, VA, 1974. Reprint of 1952 edition.