CLASSROOM CAPSULES

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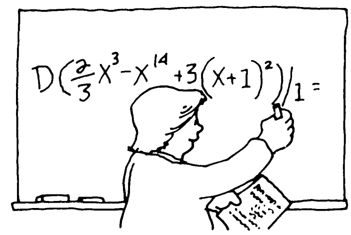

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

Summation by Parts

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This discrete analog of integration by parts, once well known to students of actuarial mathematics [1] and difference equations [2], recently reappeared in this *Journal* in an article by Lee Zia [3] to sum powers of integers. In this Capsule we will examine this and other applications of summation by parts to topics in calculus and number theory.

The actual summation by parts formula may take many different forms. For sequences $(u_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$, Zia gives the formula as

$$\sum_{k=1}^{n} (Du_k)v_k = u_k v_k \Big|_1^{n+1} - \sum_{k=1}^{n} u_k(Dv_k),$$

where $(Du_k)_{k=1}^{\infty}$ is the sequence obtained from (u_k) by defining $Du_k = u_{k+1} - u_k$ (and similarly for Dv_k). For our purposes, we find the following form for sequences $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ more convenient:

$$\sum_{k=1}^{n} b_k(a_{k+1} - a_k) + \sum_{k=1}^{n} a_k(b_k - b_{k-1}) = a_{n+1}b_n - a_1 b_0. \qquad (*)$$

This formula is readily verified by expanding the sums and noting that the result telescopes.

A nice geometric illustration of $(*)$ for the case in which both (a_k) and (b_k) are increasing sequences of positive numbers is given in Figure 1. In this case, each sum represents a sum of areas of rectangles.

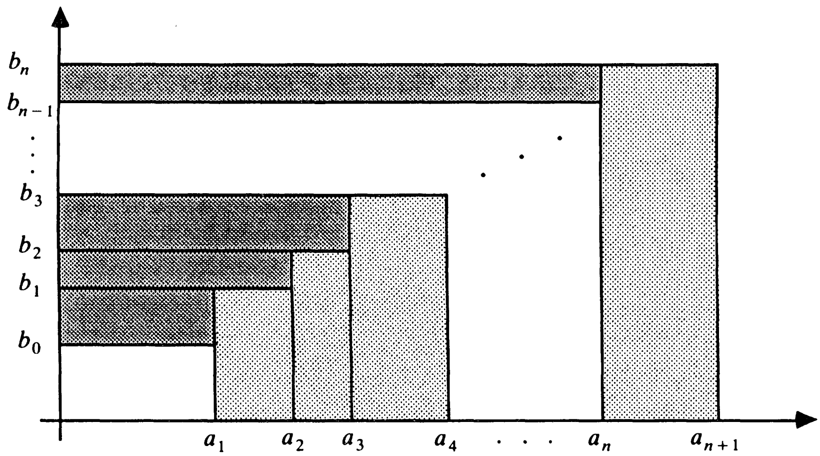

Figure 1

Sums of powers of integers. To illustrate the definition of the definite integral, it is common to do a few examples—examples that require closed expressions for the sums $\Sigma_{k=1}^{n}k$, $\Sigma_{k=1}^{n}k^2$, and $\Sigma_{k=1}^{n}k^3$. Formulas for these sums often are simply stated and then proven by induction; or derived recursively (see [**3**] and the references cited therein).

However, with appropriate choices for a_k and b_k, each of the desired expressions is an immediate consequence of $(*)$ and can be obtained without induction or recursion, and with very little algebra. If we set $a_k = b_k = k$, then each of the differences $a_{k+1} - a_k$ and $b_k - b_{k-1}$ is 1 and thus $(*)$ reduces to $2\Sigma_{k=1}^{n}k = (n+1)\cdot n$, and hence $\Sigma_{k=1}^{n}k = \frac{1}{2}n(n+1)$. If we set $a_k = k^2 - k = k(k-1)$ and $b_k = k + \frac{1}{2}$, then $(*)$ becomes

$$\sum_{k=1}^{n}\left(k+\frac{1}{2}\right)(2k) + \sum_{k=1}^{n}(k^2-k)(1) = (n+1)n\cdot\left(n+\frac{1}{2}\right),$$

so that

$$3\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{2},$$

or $\Sigma_{k=1}^{n}k^2 = \frac{1}{6}n(n+1)(2n+1)$. If we set $a_k = b_k = k^2$, then

$$\sum_{k=1}^{n}k^2(2k+1) + \sum_{k=1}^{n}k^2(2k-1) = (n+1)^2\cdot n^2,$$

and hence $\Sigma_{k=1}^{n}k^3 = \frac{1}{4}n^2(n+1)^2$.

Sequences. The standard approach to geometric series requires an expression for the sum of the first n terms from a geometric sequence. If we set $a_k = a$ and $b_k = r^k$ $(r \neq 1)$ in $(*)$, we have

$$\sum_{k=1}^{n}a(r^k - r^{k-1}) = ar^n - a,$$

so that

$$(r-1)\sum_{k=1}^{n} ar^{k-1} = a(r^n - 1),$$

and hence $\Sigma_{k=1}^{n} ar^{k-1} = a(1-r^n)/(1-r)$.

Arithmetic sequences can be dealt with similarly. We need only set $a_k = k-1$ and $b_k = a + (k-1)(d/2)$ in $(*)$ to obtain

$$\sum_{k=1}^{n}\left[a+(k-1)\frac{d}{2}\right] + \sum_{k=1}^{n}(k-1)\frac{d}{2} = n\cdot\left[a+(n-1)\frac{d}{2}\right],$$

so that $\Sigma_{k=1}^{n}[a+(k-1)d] = (n/2)[2a+(n-1)d]$.

Derivatives of powers. Using the definition to find the derivative of $f(x) = x^n$ (n a positive integer) often leads to the need to factor $x^n - c^n$, since $f'(c) = \lim_{x\to c}(x^n - c^n)/(x-c)$. Setting $a_k = x^{k-1}$ and $b_k = c^{n-k}$ in $(*)$ yields

$$\sum_{k=1}^{n} c^{n-k}(x^k - x^{k-1}) + \sum_{k=1}^{n} x^{k-1}(c^{n-k} - c^{n-k+1}) = x^n - c^n,$$

so that

$$(x-1)\sum_{k=1}^{n} c^{n-k}x^{k-1} + (1-c)\sum_{k=1}^{n} x^{k-1}c^{n-k} = x^n - c^n,$$

from which it follows that $(x-c)\Sigma_{k=1}^{n} c^{n-k}x^{k-1} = x^n - c^n$.

Reduction formulas. Just as integration by parts is often used to reduce the evaluation of one integral to the evaluation of a simpler one, summation by parts can be used in a similar fashion. We give but one example, the evaluation of $\Sigma_{k=1}^{n} k\cos k\vartheta$.

Set $a_k = \sin(2k-1)(\vartheta/2)$ and $b_k = k$. Then $b_k - b_{k-1} = 1$ and, by the product identities, $a_{k+1} - a_k = 2\cos k\vartheta \sin(\vartheta/2)$. Thus $(*)$ becomes

$$2\sin\frac{\vartheta}{2}\sum_{k=1}^{n} k\cos k\vartheta + \sum_{k=1}^{n}\sin(2k-1)\frac{\vartheta}{2} = \sin(2n+1)\frac{\vartheta}{2}\cdot n.$$

However, it is easy to show that

$$\sum_{k=1}^{n}\sin(2k-1)\vartheta = \frac{1-\cos 2n\vartheta}{2\sin\vartheta} = \frac{\sin^2 n\vartheta}{\sin\vartheta}$$

[by using $a_k = 1$ and $b_k = \cos 2k\vartheta$ in $(*)$]. It now follows that

$$\sum_{k=1}^{n} k\cos k\vartheta = \frac{n\sin(2n+1)\dfrac{\vartheta}{2}}{2\sin\dfrac{\vartheta}{2}} - \frac{\sin^2\dfrac{n\vartheta}{2}}{2\sin^2\dfrac{\vartheta}{2}}.$$

Further applications. The use of summation by parts is not limited to topics from calculus. To illustrate this, consider the sequence of *Fibonacci numbers* $(F_n) =$

$(0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Summation by parts can be used to derive a number of nice identities involving sums of Fibonacci numbers.

Let $a_k = F_{k+1}$ and $b_k = 1$ in $(*)$. Then $a_{k+1} - a_k = F_k$ and hence $(*)$ reduces to $\sum_{k=1}^{n} F_k = F_{n+2} - 1$. If we set $a_k = b_k = F_{k+1}$ in $(*)$ we have

$$\sum_{k=1}^{n} F_{k+1} F_k + \sum_{k=1}^{n} F_{k+1} F_{k-1} = F_{n+2} F_{n+1} - 1.$$

But if we combine the sums on the left, the summand simplifies to $F_{k+1}(F_k + F_{k-1}) = F_{k+1}^2$ and thus we have $\sum_{k=1}^{n} F_{k+1}^2 = F_{n+2} F_{n+1} - 1$. But since $F_1^2 = 1$, this is equivalent to $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

As a final example, set $a_k = F_{k+2}$ and $b_k = (\frac{1}{2})^k$. Then $(*)$ yields

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k F_{k+1} - \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k F_{k+2} = F_{n+3} \cdot \left(\frac{1}{2}\right)^n - 2,$$

from which it readily follows that $\sum_{k=1}^{n} (\frac{1}{2})^k F_k = 2 - (\frac{1}{2})^n F_{n+3}$.

Other identities can be similarly established. A few examples (with the choices for a_k and b_k) are indicated below:

a_k	b_k	Result
1	F_{2k}	$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$
F_{2k-2}	F_{2k}	$F_1 F_2 + F_2 F_3 + \cdots + F_{2n-1} F_{2n} = F_{2n}^2$
F_{2k-1}	F_{2k+1}	$F_1 F_2 + F_2 F_3 + \cdots + F_{2n} F_{2n+1} = F_{2n+1}^2 - 1$
F_{k+1}	k	$F_1 + 2F_2 + 3F_3 + \cdots + nF_n = nF_{n+2} - F_{n+3} + 2$
F_k	$(-1)^k$	$F_2 - F_3 + F_4 - \cdots + (-1)^n F_n = (-1)^n F_{n-1}$
$\binom{m+k}{k}$	$k+1$	$\binom{m}{0} + \binom{m+1}{1} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}$
$k+1$	$\binom{m+k}{k}$	$1\binom{m}{1} + 2\binom{m+1}{2} + \cdots + n\binom{m+n-1}{n} = m\binom{m+n}{m+1}$

[Note: The last result above appeared as Problem 1290 in *Mathematics Magazine* (February 1988, February 1989)]

We encourage the interested reader to experiment and find further applications of summation by parts.

References

1. H. Freeman, *Finite Differences for Actuarial Students*, Cambridge University Press, Cambridge, 1960.
2. S. Goldberg, *An Introduction to Difference Equations*, Wiley, New York, 1958.
3. L. Zia, Using the finite difference calculus to sum powers of integers, *College Mathematics Journal* 22 (1991) 294–300.