When $\theta_1$ and $\theta_2$ are fixed, we see from (2) that $\theta_4$ varies directly with $\theta_3$ and that $\cos\theta_4$ varies *linearly* with $\cos\theta_3$. Moreover, because of the identities

$$\sin\theta_1 \sin\theta_2 = [\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)]/2,$$
$$\cos\theta_1 \cos\theta_2 = [\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)]/2,$$
$$\cos(\theta_1 + \theta_2) = -\cos[180° - (\theta_1 + \theta_2)],$$

we also see that the "coefficients" in (2) may be expressed in terms of the maximal and minimal measures of $\theta_4$. Using these identities and letting $\theta_3 = 0°$ and $\theta_3 = 180°$ in (2) confirms the results already noted in (1). Also noteworthy is the special case when $\theta_3 = 90°$, which yields $\cos\theta_4 = \{\cos(\theta_1 - \theta_2) + \cos[180° - (\theta_1 + \theta_2)]\}/2$. Thus, when $\theta_3$ achieves the mean of its extreme values, $\cos\theta_4$ is the mean of the cosines of the extreme values of $\theta_4$.

Finally, since $\theta_4$ increases from $\theta_1 - \theta_2$ to $180° - (\theta_1 + \theta_2)$ as $\theta_3$ increases from 0° to 180°, there must be a particular measure at which $\theta_4$ and its projection $\theta_3$ are equal. Letting $\theta_4 = \theta_3 = \theta$ in (2), we find that

$$\theta = \arccos\left[\frac{\sin\theta_1 \sin\theta_2}{1 - \cos\theta_1 \cos\theta_2}\right]$$
$$= \arccos\left[\frac{\{\cos(\theta_1 - \theta_2) + \cos[180° - (\theta_1 + \theta_2)]\}/2}{1 - \{\cos(\theta_1 - \theta_2) - \cos[180° - (\theta_1 + \theta_2)]\}/2}\right].$$

Thus, the value at which $\theta_3 = \theta_4$ is always less than 90° and may be interpreted as a function of the maximal and minimal measures of $\theta_4$. For the given exercise with $\theta_1 = 37°$ and $\theta_2 = 21°$, we obtain $\theta \approx 32°$. Although these three angles sum to approximately 90°, such is not the case for $\theta_1 + \theta_2 + \theta$ in general.

## The Volume and Centroid of the Step Pyramid of Zoser

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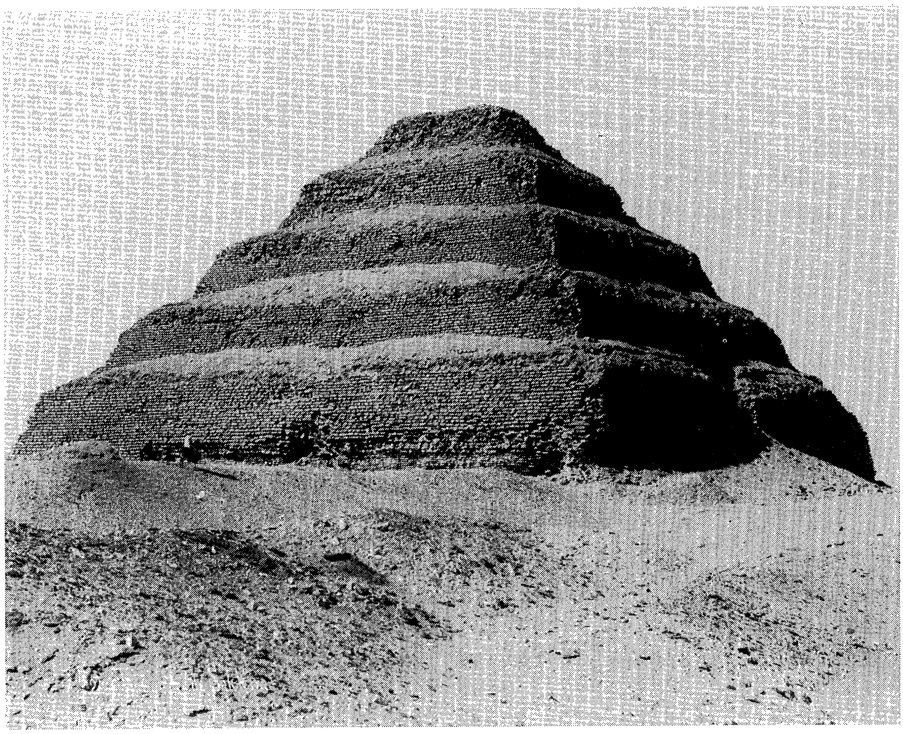
The Step Pyramid of Pharaoh Zoser at Saqqara, the most ancient of the Egyptian pyramids, was built around 2780 B.C. by the King's minister Imhotep, the official in charge of constructing a suitable sepulcher for his lord. The passage of five millennia has, of course, made Zoser's edifice a less perfect example of the ideal step pyramid than it originally was. We may also note that as the centuries progressed, the Egyptians built step pyramids with a greater number of steps, although these steps were of smaller altitude; by the time the monuments at Giza were put up, they had learned to add an external limestone incrustation to cover up the steps and make the four faces planar. However, visitors to the three pyramids of Giza will observe that this excrescence has been removed or has fallen away with time, except from the top of Chephren's structure; they will also see that the stories there are boxes with square bases.

One of the great achievements of Egyptian mathematics was the discovery of the formula for the volume of the frustum of a pyramid with square bases of length $b$ and $B$ and height $h$ [**7**, Vol. 2, pp. 2–12]:

$$V = \frac{h}{3}(b^2 + bB + B^2).$$

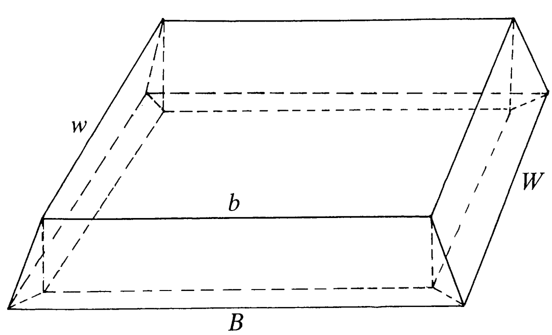


Photograph by Jean-Paul Sebah. Courtesy of The Brooklyn Museum.

This formula is found in Problem 14 of the mathematical text called the Moscow papyrus (c. 1850 B.C.). So remarkable is this achievement that Eric Temple Bell rated it above the construction of the pyramids [**4**, p. 13]. The Egyptians could have used this formula to find how much material would have been required to build a square pyramid or step pyramid of a given size.

In this capsule, we shall find the volume (and, as an additional exercise for students, the centroid) of the Step Pyramid of Zoser. Our task is a more difficult one than that of the Egyptians as the steps are not quite the frustum of a pyramid, but we have the advantage that we can use calculus.

Each slab of a step pyramid is a prismatoid having for bases two rectangles in parallel planes, and for lateral faces four trapezoids with one side lying in one base




A Prismoidal Slab of a Step Pyramid.

**Figure 1**

and the opposite side lying in the other base of the prismatoid. (See Figure 1.) We denote the dimensions of one base by $B$ and $W$, and those of the other by $b$ and $w$; we assume $B \geq b$ and $W \geq w$, and we call the height of the slab $h$. The equality $b/B = w/W$ is a necessary and sufficient condition for the slab to be a frustum of a pyramid, but, as we see from the table below, this condition is not quite satisfied by the steps of the Pyramid of Zoser.

The plane section of the slab at height $z$ from the bottom has area

$$\left[b + \left(\frac{h-z}{h}\right)(B-b)\right]\left[w + \left(\frac{h-z}{h}\right)(W-w)\right],$$

so the volume $v$ of the slab is

$$\int_0^h \left[b + \left(\frac{h-z}{h}\right)(B-b)\right]\left[w + \left(\frac{h-z}{h}\right)(W-w)\right] dz$$
$$= \frac{h}{6}(2BW + bW + Bw + 2bw), \tag{1}$$

and the total volume of the pyramid is the sum of the volumes of the six steps. (See the table.) The reader can also derive formula (1) by making the appropriate substitutions in the "prismoidal formula," which may be found, for example, in [**3**, p. 146].

The moment $m$ about the base of a slab is

$$\int_0^h z\left[b + \left(\frac{h-z}{h}\right)(B-b)\right]\left[w + \left(\frac{h-z}{h}\right)(W-w)\right] dz$$
$$= \frac{h^2}{12}(BW + bW + Bw + 3bw).$$

Once we have found $v$ and $m$, we can calculate $\bar{z}$, the height of the slab's centroid, by

$$\bar{z} = \frac{m}{v} = \tfrac{1}{2}h\left(\frac{BW + bW + Bw + 3bw}{2BW + bW + Bw + 2bw}\right). \tag{2}$$

We may note here that in the special case when the slab is a frustum of a pyramid, if we denote the areas $BW$ of the lower base and $bw$ of the upper base by $A$ and $a$ respectively, then, since $Bw = bW$ now, we have

$$v = \frac{h}{3}(A + \sqrt{Aa} + a).$$

The quantity $\frac{1}{3}(A + \sqrt{Aa} + a)$ is called the Heronian mean of $A$ and $a$, equal to the weighted mean of twice the arithmetic mean of $A$ and $a$ and the geometric mean of $A$ and $a$. (See [**5**, pp. 50–51, 168–169].) Also for the special case when the slab is a frustum, formula (2) becomes

$$\bar{z} = \tfrac{1}{4}h\left(\frac{1 + 2r + 3r^2}{1 + r + r^2}\right),$$

where $r$ stands for the common ratio $w/W = b/B$.

Now, since the pyramid consists of 6 slabs, the height of the centroid of the Step Pyramid is given by

$$\frac{\sum_{i=1}^{6} v_i(z_i + h_1 + h_2 + \cdots + h_{i-1})}{\sum_{i=1}^{6} v_i} \approx 79 \text{ feet.}$$

In the first five columns of the table, we display the dimensions of the six slabs of the Zoser pyramid as they are given (or in part calculated from the information) in [**1**, p. 382] and [**6**, II, plate 20].

We conclude by pointing out, in the words of the chief published authority on the Step Pyramid, that "the tops of the steps are covered by sloping beds of chip and dust derived from the white casing which once covered the whole structure" [**6**, I, p. 25] and that "detritus has accumulated from the weathering of the stone of the pyramidal core" [**6**, I, p. 3]. Furthermore, "the exterior of the pyramid has been so plundered and stripped of all its fine stonework that it is not easy to understand its original construction and appearance" [**6**, I, p. 3]. The reader will therefore not be surprised to learn that different authors give different measurements for the dimensions of the slabs; since even small differences in measurement will greatly affect the values of the volumes and moments of the steps (a good application of differentials), one must take the numbers presented here with a grain of salt. Also, the lowest slab of the Step Pyramid is not entirely solid, so that the center of mass of the whole structure will be higher than we have calculated. Most important of all, it is possible (probable, according to Firth) that the upper surface of each step was inclined a bit inwards so as to throw off rainwater.

**Table**

| Step | $B$ | $W$ | $b$ | $w$ | $h$ | $v$ | $m$ | $\bar{z}$ | $b/B$ | $w/W$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 398 | 354 | 376 | 332 | 37.75 | $5.02 \times 10^6$ | $9.28 \times 10^7$ | 18.5 | .946 | .939 |
| 2 | 363 | 319 | 343 | 299 | 36 | $3.93 \times 10^6$ | $6.92 \times 10^7$ | 17.6 | .943 | .935 |
| 3 | 330 | 286 | 310 | 266 | 34.5 | $3.04 \times 10^6$ | $5.13 \times 10^7$ | 16.9 | .940 | .931 |
| 4 | 297 | 253 | 278 | 234 | 32.75 | $2.29 \times 10^6$ | $3.67 \times 10^7$ | 16.0 | .937 | .926 |
| 5 | 265 | 221 | 247 | 203 | 31 | $1.69 \times 10^6$ | $2.55 \times 10^7$ | 15.1 | .933 | .920 |
| 6 | 234 | 190 | 218 | 174 | 29.33 | $1.21 \times 10^6$ | $1.72 \times 10^7$ | 14.3 | .928 | .912 |

Height of the Pyramid: 201.28 feet
Volume of the Pyramid: $1.72 \times 10^7$ cubic feet
Moment of the Pyramid about its base: $1.36 \times 10^9$
Height of the Centroid: 79 feet above the base

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## The Sum is One

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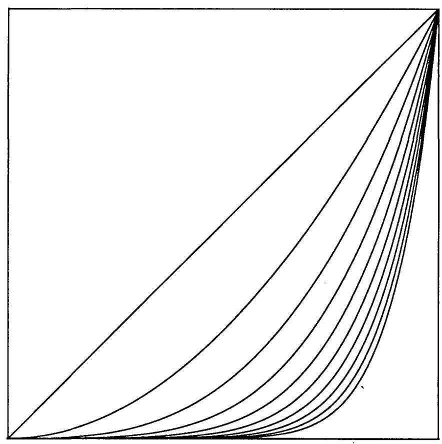
Many calculus texts contain the following telescoping series.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1. \tag{1}$$

An interesting geometric realization of this result can be obtained from the fact that the family of curves $\{x^n\}$ partitions the unit square into an infinite number of regions with total area 1. First, find the area between the curves $y = x^{n-1}$ and $y = x^n$ over the interval $[0, 1]$:

$$\int_0^1 (x^{n-1} - x^n)\, dx = \int_0^1 x^{n-1}\, dx - \int_0^1 x^n\, dx = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}. \tag{2}$$

Summing each side of (2) produces the result (1).



### What Mathematicians Believe and Some Artists Know

Science is inextricably bound up with art for Hockney. "We're living in a world that's a bit topsy-turvy. They might be very cynical about beauty at CalArts but they take it very seriously indeed at Caltech. All mathematicians take it very seriously because they react to it. In that sense, sometimes science is more interesting than what we call art now. There's a thrill to it—there's even a joy in it you don't often find in art."

Barbara Isenberg, David Hockney's New Toys, *The Los Angeles Times Calendar*, September 16, 1990.