$$= \begin{cases} 4\pi GmdR^2c^{-2} & \text{if } c > R \\ 0 & \text{if } c < R \\ 2\pi Gmd & \text{if } c = R. \end{cases}$$

The total mass of the sphere is $M(f) = 4\pi R^2 d$, and if this mass were concentrated at the center $(c, 0)$ with $c > R$, the force on the mass m at $(0, 0)$ would be $GmM(f)c^{-2}$. This is the same as the above integral, so we have a single-variable derivation of a result of Newton, that the external gravitational attraction of a sphere is equal to the attractive force of a point with the same mass at the sphere's center. This was part of Newton's argument that a solid ball has the same property.

The same integral also demonstrates the fact that if the particle of mass m is inside the sphere, so $c < R$, then it feels no force acting in any direction. (This fact was interesting and surprising to many students.) At $c = R$, the particle is on the sphere, and the force is $\frac{1}{2}GmM(f)c^{-2}$; plotting F as a function of c, there is a discontinuity at $c = R$. The $c = 0$ and $c < 0$ cases follow from similar calculations.

Other surfaces of revolution for which the above integral formula might be tractable are cylinders, $f(x) = K$, truncated cones, $f(x) = kx + K$, or funnel shapes, $f(x) = k/x$, over intervals where $f(x) \geq 0$. The construction also could be applied to a repelling force.

An Elementary Approach to e^x

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A definition of *e* Here is an elementary definition of e that lends itself to an especially quick development of the derivative of e^x: *Let e be the unique real number larger than 1 such that xe^{-x} is greatest when $x = 1$.*

In a first calculus course, graphical exploration of xa^{-x} for various values of a should be sufficient to convince students that there is such a number.

Differentiation and a classical limit The differentiability of e^x is a quick consequence of the definition. We will assume that for $a > 0$, a^x is defined and continuous on $\mathbb{R}$, and obeys standard rules of exponentiation.

By our definition of e, $xe^{-x} \leq e^{-1}$ for all x. This reduces to $x \leq e^{x-1}$ or, equivalently, $1 + x \leq e^x$ for all $x \in \mathbb{R}$. Then $1 - x \leq e^{-x}$ as well, so $e^x(1 - x) \leq 1$, and $e^x \leq 1 + xe^x$ for all x. As a result, $\frac{e^x - 1}{x}$ lies between 1 and e^x for all $x \neq 0$, the order depending on the sign of x. Now $\lim_{x\to 0} \frac{e^x-1}{x} = 1$ follows from the continuity of e^x at 0 and properties of limits. The general differentiation result is a consequence of the standard factorization $\frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h}$ and properties of limits. We conclude:

- *$f(x) = e^x$ is differentiable on $\mathbb{R}$. Furthermore, $f'(x) = e^x$.*

The limit often taken as the definition of e is also easy to obtain from our definition, without making use of the bounded monotone sequence property. The inequality $1 + x \leq e^x$ leads to $1 + \frac{1}{n} \leq e^{1/n}$ and $1 - \frac{1}{n+1} \leq e^{-1/(n+1)}$ for all positive integers n. These inequalities translate to $(1+\frac{1}{n})^n \le e \le (1+\frac{1}{n})^{n+1}$ which is enough to conclude:

- $e = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n.$

This approach leaves out some details, but so do most treatments in modern introductory calculus texts. It is an honest approach in the sense that a rigorous treatment is available at an introductory analysis level, making it a reasonable alternative to typical presentations of e^x.

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Why It Might Seem That Christmas Is Coming Early This Year

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"Daddy, how long until Christmas?" That is the question that my kids seem to ask all through the year. When I was younger, it sometimes felt that time moved very slowly, especially when I was looking forward to something such as Christmas. Now that I am an adult, time has speeded up and it seems that each year passes more quickly than the last. Now asking myself, "Is it already Christmas again?" is a regular occurrence each winter. Using the fact that at age n the current year is $1/n$ of one's life, this note presents a model of passing time that could be used as source of classroom exercises.

Let us define $D(a_1, a_2)$, the perceived duration of life from age a_1 to age a_2 by

$$D(a_1, a_2) = \sum_{n=a_1+1}^{a_2} \frac{1}{n}.$$

Take the precarious teenage years (from the 13th birthday to the 20th) as an example. People's seven teenage years may seemed to have lasted longer than their first seven years of adulthood. This can be quantified using the definition:

$$\frac{D(13,20)}{D(20,27)} = \frac{\sum_{n=14}^{20} 1/n}{\sum_{n=21}^{27} 1/n} = 1.42,$$

so the teenage years might seem to last 42% longer than the early adult years, even though both periods are of course the same length.

Since our lives are continuous and not discrete, the definition should be modified to

$$L(a_1, a_2) = \int_{a_1}^{a_2} \frac{1}{t}\,dt = \ln\left(\frac{a_2}{a_1}\right)$$

(L for length). Again comparing the teenage to early adult years we have

$$\frac{L(13,20)}{L(20,27)} = \frac{\ln(20/13)}{\ln(27/20)} = 1.44$$

which, as expected, is approximately the same as in the discrete case.

Of course, the perception of time will obviously be different for each person, depending on a number of factors. Rather than using L as a strict prediction for one person, we can look at it as giving the average perception of time for all people, similar to how the normal high temperature for a given day is an average of previous years' measurements of the high temperature on that day.

