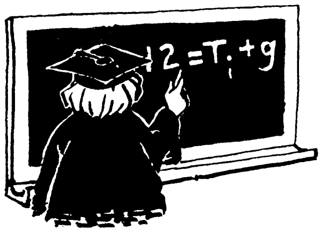


CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

Warren Page
Department of Mathematics
New York City Technical College
300 Jay Street
Brooklyn, N.Y. 11201

Reflection Property of the Ellipse and the Hyperbola

Michael K. Brozinsky, Queensborough Community College, Bayside, NY

Here is a polar coordinate proof of the known fact that *the focal radii of an ellipse (hyperbola) make equal angles with the tangent*. In other words, $\sphericalangle\alpha = \sphericalangle\beta$ in Figure 1 (Figure 2). Our proof, using a polar coordinate system at each focus, nicely combines a number of trigonometric notions and can therefore be a good review (or enrichment) project for calculus students.

Assume that focus F is p units to the left of one directrix, and focus F' is p units to the right of the ellipse's other directrix. Then the polar equations of the ellipse relative to poles F and F' are

$$r = \frac{ep}{1 + e\cos\theta}, \qquad r' = \frac{ep}{1 - e\cos\theta}, \tag{1}$$

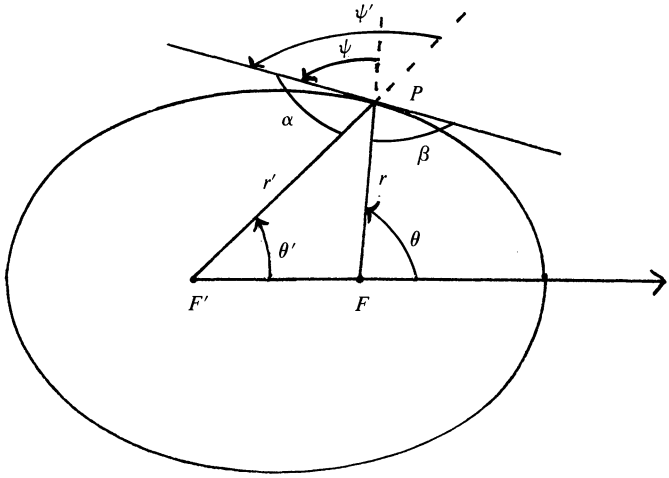
respectively. From the law of sines in triangle $F'PF$, we also have

$$2a = r + r' = \frac{2c(\sin\theta + \sin\theta')}{\sin(\theta - \theta')}. \tag{2}$$

Since $e = c/a$, one can always recast (2) as

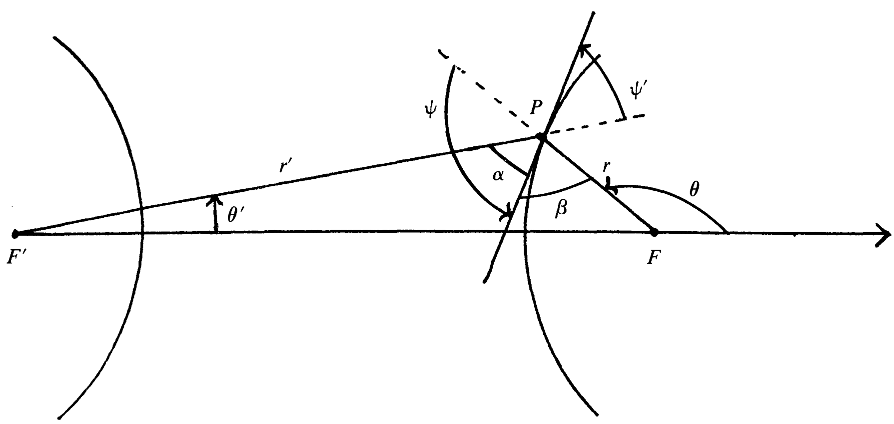
$$e = \frac{\sin(\theta - \theta')}{\sin\theta + \sin\theta'}. \tag{3}$$

Now observe that $\alpha = \beta$ if and only if $\pi - \psi' = \psi$. Since $\psi, \psi' \in [0, \pi]$, this is equivalent to the requirement that $-\tan\psi' = \tan\psi$. In polar coordinates, it is well



$r' + r = 2a \quad |F'F| = 2c$

Figure 1.



$r' - r = 2a \quad |F'F| = 2c$

Figure 2.

known that the counterclockwise angle ψ between the radius vector and the tangent line is given by $\tan\psi = \dfrac{r}{dr/d\theta}$. Thus, using (1),

$$\tan\psi = \frac{1 + e\cos\theta}{e\sin\theta}, \qquad \tan\psi' = \frac{1 - e\cos\theta'}{e\sin\theta'}. \tag{4}$$

Using the identity for $\sin(A + B)$, it is easily verified that $-\tan\psi' = \tan\psi$ is equivalent to

$$e = \frac{\sin\theta - \sin\theta'}{\sin(\theta + \theta')}. \tag{5}$$

But (5) is true by virtue of (3) and the identity

$$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B.$$

The polar coordinate proof of the hyperbola's reflection property is similar and therefore left to the reader. (Note that the branch of the hyperbola not encompassing F' is generated by negative values of r'.)

A Sequel to "Another Way of Looking at $n!$"

William Moser, McGill University, Montreal, Canada

In the *TYCMJ*'s November 1980 Classroom Capsules Column, Davis Hsu observed that $1/n!$ is the content (volume) of the n-dimensional simplex determined by

$$x_1 + x_2 + \cdots + x_n \leqslant 1, \qquad x_i \geqslant 0\ (1 \leqslant i \leqslant n).$$

Going a step further, we show that the content of the polytope P_n determined by

$$|x_1| + |x_2| + \cdots + |x_n| + |x_1 + x_2 + \cdots + x_n| \leqslant 2$$

is $\binom{2n}{n}/n!$. Figures 1 and 2 depict P_2 and P_3, respectively.

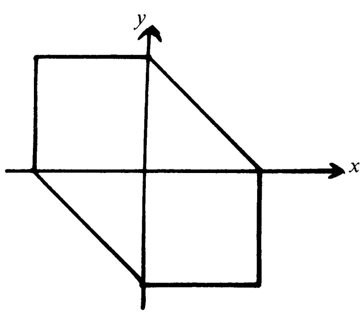


Figure 1.

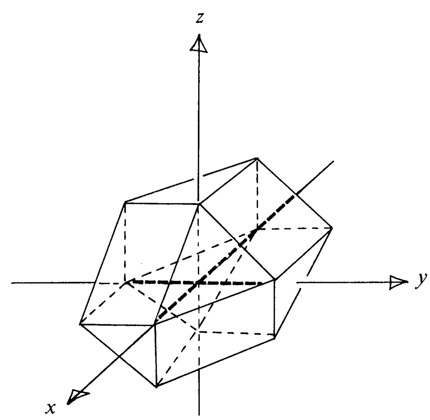


Figure 2.

In 2-space (the plane) there are 4 quadrants. In 3-space there are 8 octants. In n-space there are 2^n "octants." Indeed, for any k $(0 \leqslant k \leqslant n)$ there are $\binom{n}{k}$ octants . . . one for each choice of the k coordinates that are nonnegative (the other $n - k$ coordinates are nonpositive). For example, one such octant with k nonnegative and $n - k$ nonpositive coordinates is determined by

$$x_i \geqslant 0 \quad (1 \leqslant i \leqslant k) \quad \text{and} \quad x_i \leqslant 0 \quad (k + 1 \leqslant i \leqslant n). \tag{1}$$