Therefore, $m^{m-n} < (m/n)^m$ and $m^{n-m} > (n/m)^m$, so that $m^n > n^m$. It is of interest to note that a special case of the first part of (i) is the familiar result $e^\pi > \pi^e$.

To establish (ii), let $a = 1$. Then (1) becomes

$$(b-1)/b < \log b < b - 1. \tag{2}$$

Using the left-hand inequality in (2), we get $1 - 1/b < \log b$, or $1/b - 1 > -\log b = \log 1/b$. Since $b > 1$, it follows that $1/b \in (0,1)$. Combining this with the right-hand inequality in (2), we have shown that for any $x > 0$:

$$x - 1 \geqslant \log x, \text{ with equality holding if and only if } x = 1. \tag{3}$$

Now let

$$a = (1/n)\sum_{i=1}^{n} a_i \quad \text{and} \quad g = \sqrt[n]{a_1 a_2 \cdots a_n}\,.$$

Applying (3) to each a_i/g and adding, we obtain

$$(1/g)\left(\sum_{i=1}^{n} a_i\right) - n \geqslant \log\left[(a_1 a_2 \cdots a_n)/g^n\right],$$

or

$$(na)/g - n \geqslant 0.$$

Thus, $a \geqslant g$. Here, equality holds if and only if each $x = a_i/g$ in (3) equals 1; that is, if and only if $a_1 = a_2 = \cdots = a_n$.

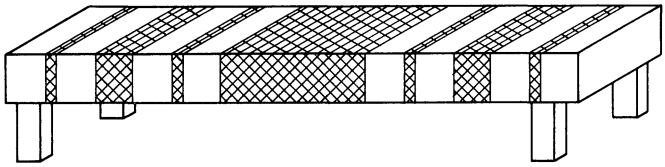
Finally, to establish (iii), put $b = (n+1)/n$ in (2) to obtain $1/(n+1) < \log(n+1)/n < 1/n$. Thus, $\log(1 + 1/n)^n < 1 < \log(1 + 1/n)^{n+1}$, which gives (iii).

Cantor's Disappearing Table

Larry E. Knop, Hamilton College, Clinton, NY

It is useful occasionally to remind students that mathematics is more than formulas and rote procedures, and that even simple ideas may have logical consequences which confound intuition. The following graphic demonstration is based on properties of a "fat" Cantor set, and the only background required is limits of sequences and the sum of a geometric series.

To begin, choose a table (or any object) having finite length l. Challenge students to make the table disappear by removing exactly half of it. Intuition (freshman intuition) says this is impossible. To show that it is not, mark the midpoint of the table and remove the section of length $l/4$ centered at the midpoint. (A piece of chalk applied to the edge of the table graphically shows the part of the table which is to be removed.) At this point, $1/4$ of the table has been removed and each of the remaining pieces has length less than $l/2$. Next, remove $1/8$ of the table by taking sections of length $l/16$ from the centers of the two remaining pieces. The amount of the table which has been removed is now $(1/4) + (1/8)$, and each of the remaining pieces has length less than $l/4$. Continue the process. The result is a table marked as shown, and two strings of numbers:



Amount of table removed: $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

Remaining lengths less than: $\frac{l}{2}, \frac{l}{4}, \frac{l}{8}, \frac{l}{16}, \cdots$

Since the total amount of table taken away is the sum of the above geometric series, exactly one-half of the table has been removed, as promised. On the other hand, the length of any remaining piece is no less than 0 and no greater than the limit of the sequence above—which is 0. There are no intervals of table of nonzero length left; the table has disappeared.

A moment of silence is the standard response to these conclusions. Is there really any table left? Because of the construction it does not seem worthwhile to try to identify any of the remaining points. It is easy, however, to show that there is still table present. The only requirement is an object longer than 1/4 the length of the table. (For short tables a calculus book is an excellent prop.) Simply put the object on the table and ask if it will fall. To fall, the object must go through a section that has been removed. Since the object is longer than $l/4$, it will not fit through any of the removed sections. No, the object will not fall; there is still table present and, in fact, 1/2 of the table is still there. There are simply no intervals of nonzero length left.

As a follow-up, ask students to show how a table can be made to disappear by removing 1/4 of the table. Can the same thing be done by removing 1/10 of the table? This demonstration takes only 10 minutes of class time, plus a bit of outside work by students, if desired. Students will accept the conclusions—the logic is too simple and straightforward—but they will not be comfortable with them: half the table remains, but there are no intervals of nonzero length! Mysteries lie hidden within simple ideas and there may be more to mathematics than they have imagined. Demonstrating this to students is worth the time spent.

Bernoulli's Inequality and the Number *e*

Joseph Wiener, Pan American University, Edinburg, TX

Our purpose is to give an elementary proof of the fact that $S_n = (1 + 1/n)^n$ increases and $T_n = (1 + 1/n)^{n+1}$ decreases to the same limit, without using the concept of the integral or properties of the natural logarithm.

The following remarks are essential:

(i) Since $T_n = (1 + 1/n)S_n$ and $(1 + 1/n) \to 1$ as $n \to \infty$, the sequences S_n and T_n cannot have different limits.

(ii) Since $T_n > 0$, the sequence T_n is bounded from below.

Therefore, it remains to prove that T_n is decreasing. Our argument is based on Bernoulli's inequality: for $x > -1$, $x \neq 0$ and all natural $n > 1$,

$$(1 + x)^n > 1 + nx. \qquad (*)$$